CLUSTER SET THEOREMS FOR UNIFORMLY CONVERGENT SEQUENCES OF FUNCTIONS

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1. **Introduction.** Let $f(z)$ be a complex-valued function defined in D: $\{|z|<1\}$, with values on the Riemann sphere S. At any point $e^{i\theta}$ of C: $\{|z|=1\}$, the (interior) cluster set, $C_D(f, e^{i\theta})$, is defined as follows: $\alpha\in C_D(f, e^{i\theta})$ if there exists a sequence $\{z_n\}$ in D such that $\lim_{n\to\infty} z_n=e^{i\theta}$ while $\lim_{n\to\infty} f(z_n)=\alpha$. For any point $e^{i\theta}$ of C, $C_D(f, e^{i\theta})$ is closed and nonempty. If G is a subset of D whose closure contains $e^{i\theta}$, the partial cluster set, $C_G(f, e^{i\theta})$, is defined analogously by requiring the sequence $\{z_n\}$ to lie in G. (For a more detailed introduction to the theory of cluster sets, see [1].)

In this paper we consider a sequence $\{f_n(z)\}$ of functions defined in D and converging uniformly to a function $f(z)$ in D. In §2 we consider the convergence of a sequence of cluster sets for $\{f_n(z)\}$ at $e^{i\theta}$ to the corresponding cluster set for $f(z)$ at $e^{i\theta}$. The function $f(z)$ is said to have an ambiguous point at $e^{i\theta}$ if there exist two simple arcs, K and L, in D terminating at $e^{i\theta}$ for which $C_K(f, e_{i\theta})\cap C_L(f, e^{i\theta})=\emptyset$. (The original references and statements about ambiguous points may be found in [1, p. 39].) In §3 we relate the ambiguous points of $f(z)$ to those of $\{f_n(z)\}$. In §4 we present related results for mappings from an arbitrary topological space into a compact metric space.

Let Z be a compact metric space with metric d. For any nonempty closed subset A of Z and any $\epsilon>0$, we let $A+\epsilon=\{z\in Z\colon \exists a\in A$ with $d(a, z)<\epsilon\}$. If $\{A_n\}$ is a sequence of nonempty closed subsets of Z, we say $\lim_{n\to\infty} A_n=A$ if for any $\epsilon>0$ there exists an integer N such that $A_n\subset A+\epsilon$ and $A\subset A_n+\epsilon$ whenever $n>N$. It is in this sense that we discuss the convergence of a sequence of cluster sets.

2. Our hypothesis in the following is that $\{f_n(z)\}$ is a sequence of arbitrary complex-valued functions converging uniformly in D to a function $f(z)$. For simplicity, we shall denote by $|a-b|$ the distance between a and b on S in the spherical metric.

THEOREM 1. *If $e^{i\theta}$ is any point of C, then* $\lim_{n\to\infty} C_D(f_n, e^{i\theta})=C_D(f, e^{i\theta})$, *and this convergence is uniform in $e^{i\theta}$.*

PROOF. Let $e^{i\theta}$ be an arbitrary point of C, and let $\epsilon>0$ be arbitrarily chosen. Then for some positive integer N, whenever $n>N$ and $z\in D$, $|f_n(z)-f(z)|<\epsilon/3$.

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Choose any $n>N$, and select any point α_n from $C_D(f_n, e^{i\theta})$. Then there exists a sequence $\{a_k(n)\} \subset D$ such that $\lim_{k\to\infty} a_k(n)=e^{i\theta}$ and $\lim_{k\to\infty} f_n[a_k(n)]=\alpha_n$ For some subsequence $\{a_j(n)\}$ of $\{a_k(n)\}$ there exists $\lim_{j\to\infty} f[a_j(n)]=w$, where $w\in C_D(f, e^{i\theta})$, while $\lim_{j\to\infty} f_n[a_j(n)] =\alpha_n$. Now we can find an integer J such that whenever $j>J$, both $|f[a_j(n)]-w|<\epsilon/3$ and $|f_n[a_j(n)]-\alpha_n|<\epsilon/3$. But then $|\alpha_n-w| \leqq |\alpha_n-f_n[a_j(n)]|+|f_n[a_j(n)]-f[a_j(n)]|+|f[a_j(n)]-w|<\epsilon$ whenever $j>J$, so that for $\alpha_n\in C_D(f_n, e^{i\theta})$, $n>N$, we have $\alpha_n\in C_D(f, e^{i\theta})+\epsilon$.

Now let any $w\in C_D(f, e^{i\theta})$ be chosen. Then there exists a sequence $\{a_k\}\subset D$ with $\lim_{k\to\infty} a_k=e^{i\theta}$ and $\lim_{k\to\infty} f(a_k)=w$. For each $n>N$ we can select a subsequence $\{a_k(n)\}$ of $\{a_k\}$ along which $f_n(z)$ has a limit $\alpha_n\in C_D(f_n, e^{i\theta})$, while $\lim_{k\to\infty} f[a_k(n)]=w$. A repetition of the argument above will show that if $w\in C_D(f, e^{i\theta})$, then $w\in C_D(f_n, e^{i\theta})+\epsilon$ for each $n>N$.

By our definition, $\lim_{n\to\infty} C_D(f_n, e^{i\theta})=C_D(f, e^{i\theta})$, and since N above is independent of the point $e^{i\theta}$, this convergence is uniform in $e^{i\theta}$.

An obvious modification of the proof of Theorem 1 yields

THEOREM 2. *Let G be any subset of D whose closure contains a point $e^{i\theta}$ of C. Then* $\lim_{n\to\infty} C_G(f_n, e^{i\theta})=C_G(f, e^{i\theta})$. *In particular, if G is the radius, ρ, to $e^{i\theta}$,* $\lim_{n\to\infty} C_\rho(f_n, e^{i\theta})=C_\rho(f, e^{i\theta})$.

From Theorem 2 we may state the following

COROLLARY. *If L is any simple arc in D terminating at $e^{i\theta}$ on C, then $f(z)$ has a limit γ as z approaches $e^{i\theta}$ along L if, and only if,* $\lim_{n\to\infty} C_L(f_n, e^{i\theta})=\{\gamma\}$. *In particular, $f(z)$ has a radial limit* $\lim_{r\to 1} f(re^{i\theta}) =\gamma$ *if, and only if,* $\lim_{n\to\infty} C_\rho(f_n, e^{i\theta})=\{\gamma\}$.

A point α belongs to the boundary cluster set, $C_B(f, e^{i\theta})$, for a function $f(z)$ at $e^{i\theta}$ if there exist: (i) a sequence $\{\tau_k\}$ of points on $C-\{e^{i\theta}\}$ such that $\lim_{k\to\infty} \tau_k=e^{i\theta}$; and (ii) a sequence of points $\{\omega_k\}$ with $\omega_k\in C_D(f, \tau_k)$ such that $\lim_{k\to\infty} \omega_k=\alpha$. If in (ii) we require that $\omega_k\in C_\rho(f, \tau_k)$, the radial cluster set of $f(z)$ at τ_k, then we have the definition of the radial boundary cluster set, $C_{BR}(f, e^{i\theta})$, for $f(z)$ at $e^{i\theta}$. (For details of the role these cluster sets play in the boundary behavior of functions meromorphic in D, see [1] and the paper of W. B. Woolf [3].)

THEOREM 3. *For any point $e^{i\theta}$ on C,* $\lim_{n\to\infty} C_B(f_n, e^{i\theta})=C_B(f, e^{i\theta})$, *and* $\lim_{n\to\infty} C_{BR}(f_n, e^{i\theta})=C_{BR}(f, e^{i\theta})$.

PROOF. It suffices to prove the first of these statements. Let any $\epsilon>0$ be given; for some integer N, $|f_n(z)-f(z)|<\epsilon/9$ for all z in D whenever $n>N$. Choose any integer $n>N$, and let $\alpha(n)$ be an arbitrary point of $C_B(f_n, e^{i\theta})$. Then there exists a sequence $\{\tau_k(n)\}$ on $C-\{e^{i\theta}\}$ and a sequence $\{\omega_k(n)\}$ such that $\lim_{k\to\infty} \tau_k(n) = e^{i\theta}$, $\omega_k(n) \in C_D(f_n, \tau_k)$, and $\lim_{k\to\infty} \omega_k(n) = \alpha(n)$.

From the proof of Theorem 1 we have $C_D(f_n, \tau) \subset C_D(f, \tau) + \epsilon/3$ for $n > N$ and any $\tau \in C$; thus for each value of k there exists a point $\gamma_k(n) \in C_D(f, \tau_k)$ such that $|\omega_k(n) - \gamma_k(n)| < \epsilon/3$. From the sequence $\{\gamma_k(n)\}$ we can select a convergent subsequence—which for simplicity we denote by $\{\gamma_k(n)\}$ itself—with a limit $\gamma \in C_B(f, e^{i\theta})$. There exists an integer $K(\epsilon, n)$ such that $|\omega_k(n) - \alpha(n)| < \epsilon/3$ and $|\gamma_k(n) - \gamma| < \epsilon/3$ whenever $k > K(\epsilon, n)$. Then for any $k > K(\epsilon, n)$ we may write $|\alpha(n) - \gamma| \leqq |\alpha(n) - \omega_k(n)| + |\omega_k(n) - \gamma_k(n)| + |\gamma_k(n) - \gamma| < \epsilon$, so that $\alpha(n) \in C_B(f, e^{i\theta}) + \epsilon$, or $C_B(f_n, e^{i\theta}) \subset C_B(f, e^{i\theta}) + \epsilon$ for $n > N$.

We wish to show also that for $n > N$ $C_B(f, e^{i\theta}) \subset C_B(f_n, e^{i\theta}) + \epsilon$. Let α be an arbitrary point of $C_B(f, e^{i\theta})$. Then for a sequence $\{\tau_k\}$ on $C - \{e^{i\theta}\}$ with $\lim_{k\to\infty} \tau_k = e^{i\theta}$ there is a sequence $\{\omega_k\}$ such that $\omega_k \in C_D(f, \tau_k)$ and $\lim_{k\to\infty} \omega_k = \alpha$.

For $n > N$ and any $\tau \in C$ we have $C_D(f, \tau) \subset C_D(f_n, \tau) + \epsilon/3$. Thus for fixed $n > N$ and each k we may select a point $\gamma_k(n) \in C_D(f_n, \tau_k)$ such that $|\omega_k - \gamma_k(n)| < \epsilon/3$. From this point on the argument repeats that above to show that $C_B(f, e^{i\theta}) \subset C_B(f_n, e^{i\theta}) + \epsilon$ for $n > N$. Hence $\lim_{n\to\infty} C_B(f_n, e^{i\theta}) = C_B(f, e^{i\theta})$ for any point $e^{i\theta}$ on C.

Similar statements of convergence can be made in terms of other types of cluster sets at a point on C.

3. **Ambiguous points.** If $\{f_n(z)\}$ converges uniformly in D to $f(z)$, it is an easy consequence of Theorem 2 that each ambiguous point of $f(z)$ on C is an ambiguous point for all but finitely many functions $f_n(z)$. Thus a function defined in D having an ambiguous point on C cannot be uniformly approximated in D by functions having no ambiguous points.

However, the limit of a uniformly convergent sequence of functions, each with an ambiguous point on C, need not have an ambiguous point. As a simple example, define a sequence $\{f_n(z)\}$, where $f_n(z) = z$ for $z \in D - K - L$, $f_n(z) = e^{i\theta} + 1/n$ for $z \in K$, $f_n(z) = e^{i\theta} - 1/n$ for $z \in L$, where $n = 1, 2, 3, \cdots$ and K, L are simple arcs in D terminating at $e^{i\theta}$ on C. The sequence converges uniformly in D to a function $f(z)$, with $f(z) = z$ for $z \in D - K - L$, $f(z) = e^{i\theta}$ for $z \in K \cup L$. For each n, $e^{i\theta}$ is an ambiguous point of $f_n(z)$, but $f(z)$ has no ambiguous points.

If we assign a crude measure to the extent to which a point of C is ambiguous for a function in D, we can relate the points which are "uniformly" ambiguous for the uniformly convergent sequence $\{f_n(z)\}$ and the ambiguous points of their limit $f(z)$. Let us say that

$f(z)$ is δ-ambiguous at $e^{i\theta}$ if there exist simple arcs K and L in D terminating at $e^{i\theta}$ such that for each $\alpha \in C_K(f, e^{i\theta})$ and each $\beta \in C_L(f, e^{i\theta})$, $|\alpha-\beta| \geqq \delta > 0$.

THEOREM 4. *If for all n and some $\delta > 0$ $f_n(z)$ is δ-ambiguous at $e^{i\theta}$, then $f(z)$ is ambiguous at $e^{i\theta}$.*

PROOF. Let ρ be chosen, $0<\rho<\delta$, and let ϵ be chosen, $0<\epsilon<\delta-\rho$. For some integer N and all z in D, $|f_n(z)-f(z)| < \epsilon/4$ when $n>N$. Select any $n>N$. For this n there exist simple arcs, $K=K(n)$ and $L=L(n)$, in D terminating at $e^{i\theta}$ such that $|\alpha_n-\beta_n| \geqq \delta$ for all $\alpha_n \in C_K(f_n, e^{i\theta})$, $\beta_n \in C_L(f_n, e^{i\theta})$.

Let α, β be arbitrarily chosen from $C_K(f, e^{i\theta})$, $C_L(f, e^{i\theta})$, respectively. We show that $|\alpha-\beta| \geqq \rho$. For some sequences $\{a_j\} \subset K$, $\{b_j\} \subset L$, we have $\lim_{j\to\infty} f(a_j)=\alpha$, $\lim_{j\to\infty} f(b_j)=\beta$. From these sequences we can select subsequences $\{a_j'\}$, $\{b_j'\}$ such that $\lim_{j\to\infty} f_n(a_j') =\gamma \in C_K(f_n, e^{i\theta})$, $\lim_{j\to\infty} f(a_j')=\alpha$, and $\lim_{j\to\infty} f_n(b_j')=\lambda \in C_L(f_n, e^{i\theta})$, $\lim_{j\to\infty} f(b_j')=\beta$.

We can find an integer J such that for $j>J$, $|f_n(a_j')-\gamma| < \epsilon/8$, $|f_n(b_j') - \lambda| < \epsilon/8$, $|f(a_j') - \alpha| < \epsilon/8$, $|f(b_j') - \beta| < \epsilon/8$. Then for $j>J$: $|\alpha-\gamma| \leqq |\alpha-f(a_j')| + |f(a_j')-f_n(a_j')| + |f_n(a_j')-\gamma| < \epsilon/2$; and $|\beta - \lambda| \leqq |\beta - f(b_j')| + |f(b_j') - f_n(b_j')| + |f_n(b_j') - \lambda| < \epsilon/2$. Now $|\lambda-\gamma| \geqq \delta$ by hypothesis, so $\delta \leqq |\lambda-\gamma| \leqq |\lambda-\beta| + |\beta-\alpha| + |\alpha-\gamma| < \epsilon + |\alpha-\beta| < (\delta-\rho) + |\alpha-\beta|$, and $|\alpha-\beta| > \rho$. Consequently, $f(z)$ is ambiguous at $e^{i\theta}$.

4. Let X be an arbitrary topological space and Z be a compact metric space with metric ρ. For each $x \in X$ denote by $\mathfrak{U}_x$ the collection of open sets in X containing x. For any nonempty subset T of X let f be any mapping of T into Z. J. D. Weston [2] defined the cluster set of f at a point $t \in T$ to be $C(f; t) = \bigcap^* \mathrm{Cl}[f(U)]$, where $\bigcap^*$ represents the intersection over all $U \in \mathfrak{U}_t$, and $\mathrm{Cl}(A)$ denotes the closure of A. For any f mapping T into Z and any $t \in T$, we see that $C(f; t)$ is nonempty and closed.

We state for reference the following lemma [2, p. 436].

LEMMA. *Let $t \in T$ and K be a compact set in Z. Suppose that, corresponding to each $U \in \mathfrak{U}_t$, a closed set $F(U)$ in Z is prescribed so that:* (i) *if $U_1 \subset U_2$, then $F(U_1) \subset F(U_2)$*; (ii) *$K \cap [\bigcap^* F(U)] = \emptyset$. Then there exists at least one $U \in \mathfrak{U}_t$ such that $K \cap F(U) = \emptyset$.*

Let $\{f_n\}$ be a sequence of mappings of T into Z which converges uniformly on T to a mapping f. That is, given any $\epsilon > 0$ there exists integer N such that whenever $n>N$, $\rho[f_n(t), f(t)] < \epsilon$ for all $t \in T$.

THEOREM 5. *For any point* $t \in T$, $\lim_{n\to\infty} C(f_n; t) = C(f; t)$, *and this convergence is uniform in t.*

PROOF. Let $\epsilon > 0$ be given. Then for some integer N, whenever $n > N$, $\rho[f_n(t), f(t)] < \epsilon/4$ for all $t \in T$. Choose any $s \in T$; suppose for some integer $n > N$ that there exists $\alpha_n \in C(f_n; s)$ such that $\rho(\alpha_n, \alpha) \geqq \epsilon$ for all $\alpha \in C(f; s)$.

Let $K = \{z \in Z: \rho(\alpha_n, z) \leqq \epsilon/2\}$; K is compact and $K \cap C(f; s) = \emptyset$. Using the lemma with $F(U) = \mathrm{Cl}[f(U)]$, we have $K \cap \mathrm{Cl}[f(U)] = \emptyset$ for some $U \in \mathfrak{U}_s$. For each $t \in U$, $\rho[\alpha_n, f(t)] \geqq \epsilon/2$. But since $\alpha_n \in C(f_n; s)$, $\alpha_n \in \mathrm{Cl}[f_n(U)]$, and we can find some $t' \in U$ with $f_n(t') \neq \alpha_n$ and $\rho[\alpha_n, f_n(t')] < \epsilon/4$. Now $\epsilon/2 \leqq \rho[\alpha_n, f(t')] \leqq \rho[\alpha_n, f_n(t')] + \rho[f_n(t'), f(t')] < \epsilon/2$. Thus for $n > N$ and any $\alpha_n \in C(f_n; s)$, there must be some $\alpha \in C(f; s)$ such that $\rho(\alpha_n, \alpha) < \epsilon$, and for $n > N$ and any $t \in T$ we have $C(f_n; t) \subset C(f; t) + \epsilon$.

Now choose any $s \in T$ and suppose there exists $n > N$ and $\alpha \in C(f; s)$ for which $\rho(\alpha, \alpha_n) \geqq \epsilon$ for any $\alpha_n \in C(f_n; s)$. If $K = \{z \in Z: \rho(\alpha, z) \leqq \epsilon/2\}$ and $F(U) = \mathrm{Cl}[f(U)]$, then $K \cap C(f_n; s) = \emptyset$, and the Lemma gives us a set $U \in \mathfrak{U}_s$ for which $K \cap \mathrm{Cl}[f(U)] = \emptyset$. Hence for each $t \in U$, $\rho[\alpha, f_n(t)] \geqq \epsilon/2$. Since $\alpha \in C(f; s)$, we can find $t' \in U$ with $f(t') \neq \alpha$, $\rho[f(t'), \alpha] < \epsilon/4$, and again we have $\epsilon/2 \leqq \rho[f_n(t'), \alpha] \leqq \rho[f_n(t'), f(t')] + \rho[f(t'), \alpha] < \epsilon/2$. Thus for any $n > N$ and any $\alpha \in C(f; s)$, there exists $\alpha_n \in C(f_n; s)$ such that $\rho(\alpha, \alpha_n) < \epsilon$, so that for $n > N$ and all $t \in T$, $C(f; t) \subset C(f_n; t) + \epsilon$.

Therefore, $\lim_{n\to\infty} C(f_n; t) = C(f; t)$, and since N is independent of $t \in T$, this limit is uniform in t.

If t is a point of T, let A be any subset of T such that $t \in \mathrm{Cl}(A)$. As a generalization of the boundary cluster set, Weston [2] defined the cluster set $C^A(f; t) = \bigcap^* \{\mathrm{Cl}[M(f; U; A)]\}$, where for each $U \in \mathfrak{U}_t$ $M(f; U; A) = \bigcup_{A \cap U} C(f; a)$. In addition, as a generalization of the partial cluster set, let $C_A(f; t) = \bigcap^* \mathrm{Cl}\, f(A \cap U)$. Then simple modifications in the proof of Theorem 5 will yield

THEOREM 6. $\lim_{n\to\infty} C^A(f_n; t) = C^A(f; t)$ *and* $\lim_{n\to\infty} C_A(f_n; t) = C_A(f; t)$.

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