FINITE SUMS OF IRREDUCIBLE FUNCTIONALS ON C^*-ALGEBRAS[1]

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Let $\mathfrak{A}$ be a C^*-algebra with identity I and let f be a positive functional on $\mathfrak{A}$. If $L(f)$ is the closed left ideal in $\mathfrak{A}$ defined by $\{A \in \mathfrak{A} | f(A^*A)=0\}$, we may define two norms on the left $\mathfrak{A}$-module $\mathfrak{A}-L(f)$: (1) $\|\mathfrak{A}-L(f)\|_1 = f(A^*A)^{1/2}$, and (2) $\|A-L(f)\|_2 = \inf\{\|A-K\| \,|\, K \in L(f)\}$. The completion $H(f)$ of $\mathfrak{A}-L(f)$ under the first norm is a Hilbert space. The inner product on $H(f)$ is the extension of $(A-L(f), B-L(f)) = f(B^*A)$ to $H(f)$. The algebra $\mathfrak{A}$ can be represented as a C^*-algebra on the Hilbert space $H(f)$. The operator $\Phi_f(A)$ $(A \in \mathfrak{A})$ on $H(f)$ is the extension of

$$\Phi_f(A)(B - L(f)) = AB - L(f) \quad \text{to} \quad H(f).$$

The homomorphism Φ_f of $\mathfrak{A}$ into the algebra of bounded linear operators on $H(f)$ is called the canonical representation of $\mathfrak{A}$ on $H(f)$ induced by f. Under the second norm $\mathfrak{A}-L(f)$ is a Banach space which we shall denote by $X(f)$.

The two norms are related by the inequality

$$\|A - L(f)\|_1 \leqq f(I)^{1/2}\|A - L(f)\|_2,$$

for all A in $\mathfrak{A}$. Thus the map $A-L(f)\to A-L(f)$ of $X(f)$ into $H(f)$ is continuous. If $\mathfrak{A}-L(f)$ is complete under the first norm (i.e. if $H(f) = \mathfrak{A}-L(f)$), there is a bicontinuous isomorphism of $H(f)$ onto $X(f)$. In this case we say $H(f)$ is equivalent to $X(f)$ and write $H(f) \sim X(f)$. If $H(f)$ is equivalent of $X(f)$, there is a $\beta>0$ such that

$$\|A - L(f)\|_1 \leqq f(I)^{1/2}\|A - L(f)\|_2 \leqq \beta f(I)^{1/2}\|A - L(f)\|_1,$$

for all A in $\mathfrak{A}$.

Kadison [2] showed that $H(f)\sim X(f)$ for every irreducible functional f on $\mathfrak{A}$. In the present paper we prove that $H(f)\sim X(f)$ if and only if f is the finite sum of irreducible functionals.

PROPOSITION 1. *Let $\mathfrak{A}$ be a C^*-algebra with identity I and let $g_1, g_2, \cdots, g_n$ be irreducible functionals on $\mathfrak{A}$. If $f = \sum\{g_j,\ 1 \leqq j \leqq n\}$, we have $X(f)\sim H(f)$.*

PROOF. We divide the proof into three parts.

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I. Suppose $L(g_k)$ does not contain the left ideal $\bigcap_{j\neq k} L(g_j)$ for each $k=1, 2, \cdots, n$. There is a B_k in $\bigcap_{j\neq k} L(g_j)$ such that $B_k \notin L(g_k)$ for each $k=1, 2, \cdots, n$. We have $B_k - L(g_k)$ is not zero in $H(g_k)$. Because $\Phi_{g_k}(\mathfrak{A})$ is irreducible on $H(g_k)$ there is an element R_k in A such that $I-L(g_k)=\Phi_{g_k}(R_k)(B_k-L(g_k))=R_kB_k-L(g_k)$. We have $R_kB_k \in \bigcap_{j\neq k} L(g_j)$. So there is no loss of generality in assuming $B_k \equiv I \pmod{L(g_k)}$ for each $k=1, 2, \cdots, n$.

We now show $\mathfrak{A}-L(f)$ is complete under the norm $\|\cdot\|_1$. Let $\{A_m-L(f)\}$ be a Cauchy sequence in $H(f)$. Then

$$\lim_{m,p} f((A_m - A_p)^*(A_m - A_p)) = 0.$$

Therefore,

$$\lim_{m,p} g_j((A_m - A_p)^*(A_m - A_p)) = 0 \qquad \text{for } j = 1, 2, \cdots, n.$$

Since $X(g_j)\sim H(g_j)$, there is a C_j in $\mathfrak{A}$ such that

$$\lim_{m} g_j((A_m - C_j)^*(A_m - C_j)) = 0 \qquad (1 \leqq j \leqq n).$$

Let $D=\sum_{j=1}^n C_jB_j$. We claim $D-L(f)$ is the limit of $(A_m-L(f))$ in $H(f)$. Indeed, we have

$$\begin{aligned}\lim_m f((A_m - D)^*(A_m - D)) &= \sum_{j=1}^n \lim_m g_j((A_m - D)^*(A_m - D)) \\ &= \sum_{j=1}^n \lim_m g_j\left(\left(A_m - \sum_k C_kB_k\right)^*\left(A_m - \sum_k C_kB_k\right)\right) \\ &= \sum_{j=1}^n \lim_m g_j((A_m - C_j)^*(A_m - C_j)) = 0,\end{aligned}$$

since

$$g_j\left(\left(A_m - \sum_k C_kB_k\right)^*\left(A_m - \sum_k C_kB_k\right)\right) = g_j((A_m - C_jB_j)^*(A_m - C_jB_j)).$$

This proves our claim. Hence, in case I we have $X(f)\sim H(f)$.

II. Now let h be a positive functional on $\mathfrak{A}$ such that $X(h)\sim H(h)$. Let g be a second positive functional with $L(g)\supset L(h)$. We have $X(g+h)\sim H(g+h)$. Indeed, let $\beta>0$ be a scalar with the property

$$h(A^*A) \leqq h(I) \inf\{\|A - K\|^2 \mid K \in L(h)\} \leqq \beta h(I)h(A^*A),$$

for all A in $\mathfrak{A}$. Then we have for all A in $\mathfrak{A}$ that

$$
\begin{aligned}
h(A^*A) + g(A^*A) &\leqq (h(I) + g(I)) \inf\{\|A - K\|^2 \mid K \in L(g+h)\} \\
&= (h(I) + g(I)) \inf\{\|A - K\|^2 \mid K \in L(h)\} \\
&\leqq \beta(h(I) + g(I))h(A^*A) \\
&\leqq \beta(h(I) + g(I))(h(A^*A) + g(A^*A)),
\end{aligned}
$$

because $L(g+h)=L(g)\cap L(h)=L(h)$. So $X(g+h)\sim H(g+h)$.

III. We are now ready to prove Proposition 1 by induction on the number n of irreducible functionals which appear in the sum expression for f.

Let $n=2$; consider the left ideals $L(g_1)$ and $L(g_2)$. If $L(g_1)\supset L(g_2)$ or $L(g_2)\supset L(g_1)$, we may apply part II to obtain $X(g_1+g_2)\sim H(g_1+g_2)$. If, on the other hand, neither $L(g_1)\supset L(g_2)$ nor $L(g_2)\supset L(g_1)$ is true, we may apply part I to obtain $X(g_1+g_2)\sim H(g_1+g_2)$.

Let us assume that for any $k<n$ the sum h of k irreducible functionals has the desired property $X(h)\sim H(h)$. Let us show that $f=g_1+g_2+\cdots+g_n$ satisfies $X(f)\sim H(f)$ if the functionals g_j are irreducible. We have either

(a) $L(g_k)\not\supset\bigcap_{j\neq k} L(g_j)$ for $k=1, 2, \cdots, n$ or

(b) there is a k, $1\leqq k\leqq n$, with

$$L(g_k) \supset \bigcap_{j\neq k} L(g_j).$$

In case (a) the proof follows from part I. The induction hypothesis does not come into play. In case (b) we have $g_1+\cdots+g_{k-1}+g_{k+1}+\cdots+g_n=h$ has the property $X(h)\sim H(h)$ by the induction hypothesis. Also $L(g_k)\supset L(h)=\bigcap_{j\neq k} L(g_j)$. By part II, $X(f)=X(h+g_k)\sim H(h+g_k)=H(f)$. Q.E.D.

The converse to Proposition 1 is contained in Proposition 2. We prove a preliminary lemma to clarify the meaning of equivalence between $H(f)$ and $X(f)$.

Lemma 1. *Let $\mathfrak{A}$ be a C^*-algebra with identity I and let f be a positive functional on $\mathfrak{A}$ such that $H(f)\sim X(f)$. The canonical representation $\Phi=\Phi_f$ of $\mathfrak{A}$ on $H(f)$ induced by f produces an algebra $\Phi(\mathfrak{A})$ whose commutator $\Phi(\mathfrak{A})'$ on $H(f)$ has the following property: if F' is a nonzero projection in $\Phi(\mathfrak{A})'$ then F' majorizes a minimal projection E' of $\Phi(\mathfrak{A})'$.*

Proof. We may assume that $\Phi(\mathfrak{A})'$ does not consist of scalar multiples of the identity. Let F' be a nonzero projection in $\Phi(\mathfrak{A})'$. There is no loss of generality in assuming $F'\neq I$. Let $G'=I-F'$ and let $L=\{A\in\mathfrak{A} \mid A-L(f)\in G'(H(f))\}$. We have $L\supset L(f)$ and $L-L(f)=G'(H(f))$ because $H(f)\sim X(f)$. We prove L is a closed left ideal in $\mathfrak{A}$.

Indeed, L is obviously a linear manifold in $\mathfrak{A}$. If $A\in\mathfrak{A}$, $B\in L$, we have

$$\begin{aligned} AB - L(f) &= \Phi(A)(B - L(f)) \\ &= \Phi(A)G'(B - L(f)) = G'(AB - L(f)). \end{aligned}$$

Thus, $AB\in L$ and L is a left ideal. If $\{A_n\}$ is a sequence of elements in L and if $\lim_n A_n = A$, we have

$$\begin{aligned} G'(A - L(f)) &= G'\left(\lim_n (A_n - L(f))\right) \\ &= \lim_n G'(A_n - L(f)) = \lim_n A_n - L(f) = A - L(f). \end{aligned}$$

This shows that L is a closed set in $\mathfrak{A}$ and hence L is a closed left ideal in $\mathfrak{A}$. Because $I\notin L$, L is a proper ideal.

There is a maximal left ideal L_1 in $\mathfrak{A}$ such that $L_1\supset L$. Consider the set $L_1 - L(f)$ in $H(f)$. This set is obviously a linear manifold invariant under $\Phi(\mathfrak{A})$. By the equivalence of $H(f)$ and $X(f)$ and by the fact that L_1 is closed in $\mathfrak{A}$, we have that $L_1 - L(f)$ is a closed subspace in $H(f)$. Let E_1' be the projection on $H(f)$ corresponding to $L_1 - L(f)$. Because $L_1 - L(f)$ is invariant under $\Phi(\mathfrak{A})$, we have that E_1' is a member of $\Phi(\mathfrak{A})'$.

We have that $E_1'G'(A - L(f)) = E_1'(B - L(f)) = B - L(f) = G'(A - L(f))$ where $B\in L$. Thus, $E_1'\geqq G'$ or equivalently $E' = I - E_1' \leqq F'$. Since $I\notin L_1$, E_1' is not the identity operator I' on $H(f)$. It is therefore sufficient to show that the nonzero projection E' is a minimal projection in $\Phi(\mathfrak{A})'$. We suppose E' is not minimal in $\Phi(\mathfrak{A})'$ and obtain a contradiction. If E_2', E_3' are nonzero orthogonal projections in $\Phi(\mathfrak{A})'$ such that $E' = E_2' + E_3'$, we have $I' > E_2' + E_1' > E_1'$. The set $L_2 = \{A\in\mathfrak{A} \mid A - L(f)\in(E_1' + E_2')(H(f))\}$ is a proper closed left ideal containing L_1 but not equal to L_1. This follows from the previous work. Since L_1 is a maximal ideal, this is impossible. Therefore, E' is indeed a minimal projection. Q.E.D.

PROPOSITION 2. *Let $\mathfrak{A}$ be a C^*-algebra with identity I and let f be a positive functional on $\mathfrak{A}$ such that $H(f)\sim X(f)$. Then f is the sum of a finite number of irreducible functionals.*

PROOF. Let $\Phi = \Phi_f$ be the canonical representation of $\mathfrak{A}$ on $H(f)$ induced by f. Let $\{E_n'\}$ be a maximal orthogonal set of minimal projections in $\Phi(\mathfrak{A})'$. Since every nonzero projection in $\Phi(\mathfrak{A})'$ majorizes a minimal projection, we have that the least upper bound of the set $\{E_n'\}$ is $I' = \Phi(I)$. The vector $I - L(f)$ is cyclic under $\Phi(\mathfrak{A})$ and, therefore, $I - L(f)$ is a separating vector for $\Phi(\mathfrak{A})'$. This implies that

$\{E_n'\}$ is a sequence. We have that the functional f is the weak limit of functionals f_n, $n=1, 2, \cdots$, where

$$f_n = \sum \{w_{E_j'(I-L(f))}\cdot\Phi \mid 1 \leqq j \leqq n\}.$$

We shall prove that each $g_j = w_{E_j'(I-L(f))}\cdot\Phi$, $j=1, 2, \cdots$, is irreducible and that for some integer N, $0=f_N=f_{N+1}= \cdots =f_{N+k}$ for all $k>0$.

We first prove that $w_{E'(I-L(f))}\cdot\Phi = g$ is irreducible whenever E' is a minimal projection in $\Phi(\mathfrak{A})'$. Let L be the closed left ideal in $\mathfrak{A}$ given by $L=\{A\in\mathfrak{A} \mid A-L(f)\in(I-E')(H(f))\}$. Since E' is a minimal projection, the ideal L is a maximal left ideal in $\mathfrak{A}$. We have that $L\subset L(g)$ and so $L=L(g)$. There is an irreducible functional g' such that $L=L(g')$. For all A in $\mathfrak{A}$ we have

$$\begin{aligned} g(A^*A) &\leqq g(I)\inf\{\|A-K\|^2 \mid K\in L(g)\} \\ &\leqq g(I)\inf\{\|A-K\|^2 \mid K\in L(g')\} \leqq \alpha g(I)g'(A^*A), \end{aligned}$$

for some fixed scalar $\alpha>0$. Since $g(I)\neq 0$, there is $\beta>0$ such that $\beta g'(A)=g(A)$ for all A in $\mathfrak{A}$. This shows that g is irreducible on $\mathfrak{A}$.

To complete the proof we assume that the sequence $\{E_n'\}$ is not finite and arrive at a contradiction. Let

$$E_n'(I-L(f)) = E_n - L(f) \qquad \text{for each } n = 1, 2, \cdots.$$

Here $E_n\in\mathfrak{A}$. We have

$$\begin{aligned} E_n - L(f) &= E_n'(E_n - L(f)) = E_n'(\Phi(E_n)(I-L(f)) \\ &= \Phi(E_n)[E_n'(I-L(f))] \\ &= \Phi(E_n)[E_n - L(f)] = E_n^2 - L(f). \end{aligned}$$

Thus, for each n we have $E_n - E_n^2 \in L(f)$. Now let $\alpha>0$ be a scalar such that for all $A\in\mathfrak{A}$ we have

$$\inf\{\|A-K\| \mid K\in L(f)\} \leqq \alpha^{1/2}f(A^*A)^{1/2}.$$

We have

$$\|E_n - L(f)\|_2^2 \leqq \alpha f(E_n^*E_n) = \alpha(E_n - L(f), E_n - L(f)), \qquad \text{for all } n.$$

Since

$$\begin{aligned} +\infty > f(I) = \|I-L(f)\|_1^2 &= \Big\|\sum_n E_n'(I-L(f))\Big\|_1^2 \\ &= \sum_n \|E_n'(I-L(f))\|_1^2 = \sum_n \|E_n - L(f)\|_1^2, \end{aligned}$$

we have

$$\lim_n \|E_n - L(f)\|_1^2 = 0$$

and thus

$$\lim_n \|E_n - L(f)\|_2^2 \leqq \alpha \lim_n \|E_n - L(f)\|_1^2 = 0.$$

Because $L(g_n) \supset L(f)$, we have that

$$\inf\{\|E_n - K\| \mid K \in L(g_n)\} \leqq \|E_n - L(f)\|_2 \qquad \text{for each } n.$$

So $\lim_n(\inf\{\|E_n - K\| \mid K \in L(g_n)\}) = 0$. But

$$g_n((I - E_n)^*(I - E_n)) = (\Phi(I - E_n)(E_n - L(f)),$$

$$\Phi(I - E_n)(E_n - L(f)) = (E_n - E_n^2 - L(f), E_n - E_n^2 - L(f)) = 0.$$

So $I - E_n \in L(g_n)$. However, if

$$\inf\{\|E_n - K\| \mid K \in L(g_n)\} = \inf\{\|I - K\| \mid K \in L(g_n)\} < 1,$$

we would be able to find an element K in $L(g_n)$ such that $\|I - K\| < 1$. This means that K has an inverse. This is impossible and so we have obtained a contradiction to $\lim_n \inf\{E_n - K \mid K \in L(g_n)\} = 0$. Thus, there are only a finite number of nonzero E_n'. Q.E.D.

Let $\mathfrak{A}$ be a C^*-algebra on a Hilbert space H. A vector $h_0 \in H$ will be called *strictly cyclic* under $\mathfrak{A}$ if and only if the set $\{Ah_0 \mid A \in \mathfrak{A}\}$ is equal to H. Then the preceding theorems may be rephrased to give the theorem:

Let f be a positive functional on a C^*-algebra $\mathfrak{A}$ and let Φ be the canonical representation of $\mathfrak{A}$ on $H(f)$ induced by f. Then $H(f)$ has a strictly cyclic vector under $\Phi(\mathfrak{A})$ if and only if f is equal to a finite sum of irreducible functionals on $\mathfrak{A}$.

If f is a positive functional on $\mathfrak{A}$ such that $H(f) \sim X(f)$, then $f = g_1 + g_2 + \cdots + g_n$ where $g_1, g_2, \cdots, g_n$ are irreducible functionals on A. The canonical representations Φ_{g_j} of A on $H(g_j)$ induced by g_j $(1 \leqq j \leqq n)$ are determined by f up to unitary equivalence.

PROPOSITION 3. *Let $g_1, g_2, \cdots, g_n$ and $g_1', g_2', \cdots, g_m'$ be two sets of irreducible functionals on A; if $\sum_{j=1}^n g_j = \sum_{j=1}^m g_j'$, there is for each g_j a scalar $\alpha > 0$, and an index k $(1 \leqq k \leqq m)$ such that $\alpha g_j(U^*AU) = g_k'(A)$ for all A in $\mathfrak{A}$.*

PROOF. We let $f = \sum\{g_j \mid 1 \leqq j \leqq n\} = \sum\{g_j \mid 1 \leqq j \leqq m\}$. Let $\beta > 0$

be a scalar such that $\|A-L(f)\|_2^2 \leqq \beta f(A^*A)$ for all A in $\mathfrak{A}$. We have that $L(g_j) \supset L(f)$, and therefore

$$\begin{aligned} g_j(A^*A) &\leqq g_j(I)\inf\{\|A - K\|^2 \mid K \in L(g_j)\} \\ &\leqq g_j(I)\inf\{\|A - K\|^2 \mid K \in L(f)\} \leqq \beta g_j(I) f(A^*A), \end{aligned}$$

for all A in $\mathfrak{A}$. Let Φ be the canonical representation of $\mathfrak{A}$ on $H(f)$ induced by f. There is an A' in $\Phi(\mathfrak{A})'$ such that

$$g_j(A) = (\Phi(A)A'(1 - L(f)), A'(I - L(f)),$$

for all A in $\mathfrak{A}$. Let $A'(I-L(f)) = B-L(f)$, where $B \in \mathfrak{A}$. Then $g_j(A) = f(B^*AB)$ for all A in $\mathfrak{A}$. Since $g_j \neq 0$, there is an index k $(1 \leqq k \leqq m)$ such that $g_k'(B^*B) \neq 0$. By a theorem of Glimm and Kadison [1] there is a unitary operator U in $\mathfrak{A}$ and a scalar $\gamma > 0$ such that $\gamma g_k'(U^* \cdot U) = g_k'(B^* \cdot B)$. Since $g_j(A^*A) = f(B^*A^*AB) \geqq \gamma g_k'(U^*A^*AU)$, for all A in $\mathfrak{A}$, there is an $\alpha > 0$ such that $g_j(A) = \alpha g_k'(U^*AU)$, for all $A \in \mathfrak{A}$. This completes the proof.

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