THE ESSENTIAL SET OF FUNCTION ALGEBRAS

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Let X be a compact Hausdorff space and $C(X)$ the Banach algebra of all complex-valued continuous functions on X under the sup-norm. By a function algebra A we mean a closed subalgebra of $C(X)$ which separates points and contains the constant functions. Let $S(A)$ denote the space of maximal ideals of A and let $E(A)$ denote the essential set of A as defined by Bear [1], i.e. $E(A)$ is the hull of the largest closed ideal of $C(X)$ which is contained in A. We will obtain another characterization of the essential set in the case when $S(A)=X$ and thereby obtain some results of a global nature from local hypotheses.

THEOREM 1. *Let A be a function algebra on a compact metric space X. Suppose $X=S(A)$ and let E be the essential set of A in X. If x_0 has a neighborhood V such that $A \mid \overline{V}=C(\overline{V})$ then $x_0 \notin E$.*

PROOF. Let V be a neighborhood of x such that $A \mid \overline{V}=C(\overline{V})$. Then x_0 is a local peak point, and hence [2, Theorem 4.1] x_0 is a global peak point. Let $f \in A$, $f(x_0)=1$ and $|f(x)|<1$ if $x \neq x_0$. Let r be a number so close to one that

$$U = \{x\colon \operatorname{Re} f(x) \geqq r\} \subset V.$$

Then $\operatorname{Re} f(x) \leqq s<r<1$ for all $x \in X \sim V$. Let p_n be a sequence of polynomials which converge uniformly on $f[U] \cup f[X \sim V]$ to a function one on $f[U]$ and zero on $f[X \sim V]$ (see e.g. Wermer: *Banach algebras and analytic functions*, Theorem 7.6). Then $p_n \circ f$ converges uniformly on X, except possibly on $V \sim U$, to a function one at x_0 and zero off V. Let $g \in A$, $g(x_0)=1$ and $g \equiv 0$ on $V \sim U$. Then $g(p_n \circ f)$ converges uniformly on X to a function $k \in A$ such that $k(x_0)=1$, and $k \equiv 0$ off U. Let W be a neighborhood of x such that $k \neq 0$ on W. Let h be any continuous function on X which is zero off W. Let $w \in A$ with $w=h/k$ on W. Then $h=wk \in A$. That is, A contains every continuous function zero off W. Hence $X \sim W \supset E$, and $x_0 \notin E$.

COROLLARY. *Let A be a function algebra on the compact metric space $X=S(A)$ and let E be the essential set of A in X. Then $E=X \sim P$, where $P=\{x \in X\colon A \mid \overline{V}=C(\overline{V})$ for some neighborhood V of $x\}$.*

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THEOREM 2. *Let A be a function algebra on a compact metric space $X=S(A)$. Let $F_1, F_2, \cdots$ be a sequence of closed sets such that $X=\bigcup_{i=1}^{\infty} F_i$ and $A \mid F_i$ is closed in $C(F_i)$ for $i=1, 2, \cdots$. Then $\overline{\bigcup}_{i=1}^{\infty} E_i$ is the essential set of A in X where E_i is the essential set of $A \mid F_i$ in F_i.*

PROOF. Let E denote the essential set of A in X and let I denote the largest closed ideal of $C(X)$ contained in A. Let J_i be the largest ideal of $C(F_i)$ which is contained in $A \mid F_i$. Clearly $I \mid F_i$ is contained in J_i. Thus for $i=1, 2, \cdots, E_i=\{x\in F_i: f(x)=0 \text{ for all } f \text{ in } J_i\}$ is contained in $E=\{x\in X: f(x)=0 \text{ for all } f \text{ in } I\}$. Since E is closed, $\overline{\bigcup}_{i=1}^{\infty} E_i \subset E$. Let $U=E\sim\overline{\bigcup} E_i$. It is shown in Bear [1] that $A \mid E$ is a function algebra with essential set E and maximal ideal space E. Since $U=\bigcup_{i=1}^{\infty} (U\cap F_i)$ is open in E, it follows from the Baire Category theorem that some $U\cap F_j$ has a nonempty interior in E. Let V be a closed neighborhood, relative to E, which is contained in $U\cap F_j$. Since V is disjoint from E_j, the essential set of $A \mid F_j$, it follows that $A \mid V=C(V)$. This contradicts the result of Theorem 1, namely that $P=\{x\in E: A \mid V_x=C(V_x) \text{ for some closed neighborhood } V_x \text{ of } x\}$ must be the empty set.

COROLLARY. *Let A be a function algebra on the compact metric space $X=S(A)$. Let $F_1, F_2, \cdots$ be a sequence of closed sets such that $X=\bigcup_{i=1}^{\infty} F_i$ and $A \mid F_i=C(F_i)$ for $i=1, 2, \cdots$. Then $A=C(X)$.*

If X is a union of a finite number of F_i and if $A \mid F_i=C(F_i)$, it is a necessary consequence that $S(A)=X$ as the following lemma shows.

LEMMA. *Let X be a compact Hausdorff space and let $F_1, \cdots, F_n$ be n closed sets such that $X=\bigcup_{i=1}^{n} F_i$. If A is a function algebra on X such that $A \mid F_i$ is closed in $C(F_i)$ and $S(A \mid F_i)=F_i$ for $i=1, 2, \cdots, n$ then $S(A)=X$.*

PROOF. Let M be a proper maximal ideal in A. If $M \mid F_i=A \mid F_i$ for $i=1, 2, \cdots, n$, there exist functions $f_1, \cdots, f_n$ in M such that $f_i(x)=1$ whenever x is in F_i. Let $h_2=f_1+f_2-f_1f_2$. Assuming h_j has been defined let $h_{j+1}=f_{j+1}+h_j-f_{j+1}h_j$. We thus get a function h_n in M such that h_n is one on X. This contradicts the assertion that M is a proper ideal. There must therefore exist an integer k $(k\leqq n)$ such that $M \mid F_k$ is a proper ideal in $A \mid F_k$. The ideal $M \mid F_k$ is contained in a maximal ideal of $A \mid F_k$. Thus there exists x_0 in F_k such that $M \mid F_k \subset \{f\in A \mid F_k: f(x_0)=0\}$. It then follows that M corresponds to evaluation at x_0. This completes the proof that $S(A)=X$.

THEOREM 3. *Let A be a function algebra on a compact metric space X.*

Let $F_1, \cdots, F_n$ *be* n *closed sets such that* $X=\bigcup_{i=1}^{n} F_i$ *and* $A|F_i=C(F_i)$ *for* $i=1, \cdots, n$. *Then* $A=C(X)$.

PROOF. Since $S(A)=X$ by the above lemma, this theorem is precisely the corollary after Theorem 2 in the finite case.

COROLLARY. *Let* A *be a function algebra on a compact metric space* X. *If for each* x *in* X *there exists a closed neighborhood* V_x *of* x *such that* $A|V_x=C(V_x)$, *then* $A=C(X)$.

I wish to thank the referee for supplying a proof of Theorem 1, much shorter than the proof I originally submitted.

REFERENCES

1. H. S. Bear, *Complex function algebras*, Trans. Amer. Math. Soc. **90** (1959), 383–393.

2. H. Rossi, *The local maximum modulus theorem*, Ann. of Math. **72** (1960), 1–11.

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