ON A PROBLEM OF C. E. SHANNON IN GRAPH THEORY

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1. **Introduction.** We consider in this paper only finite nondirected graphs without multiple edges and we assume that on each vertex of the graph there is a loop, i.e. each vertex of the graph is connected to itself by an edge. O. Ore in [2] raised the following problem: Given a finite graph F, what are the necessary and sufficient conditions on G in order that

$$\mu(G \times H) = \mu(G) \cdot \mu(H) \qquad \text{for every finite graph } H. \tag{1}$$

A partial answer is given by the following theorem due to Shannon [3]:

THEOREM 1 (SHANNON). *If there exists a preserving function σ defined on G such that $\sigma(G)$ is an independent set of vertices in G then* (1) *holds for every finite graph H.*

For a proof of Shannon's theorem see for example [1], [3].

Shannon proved the sufficiency of his condition only. Our main result is a necessary and sufficient condition under which (1) always holds (Theorem 2) and to show that Shannon's condition is not necessary (§4). Our condition will be given in terms of linear programming.

2. **Definitions and notations.** By an independent set of vertices in a graph G we mean a subset of vertices such that no two different vertices in the subset are joined by an edge in G. The maximal number of independent vertices in a graph G will be denoted by $\mu(G)$. A clique in a graph G is a complete subgraph of G (i.e. a set of vertices each pair of which are connected by an edge) which is not contained in any other complete subgraph of G. Ver(G) will denote the set of vertices of G. A function σ: $G \to G$ will be called preserving if $g \not\leftrightarrow g' \Rightarrow \sigma(g) \not\leftrightarrow \sigma(g')$ (where $g \not\leftrightarrow g'$ means that g is not joined by an edge to the vertex g'). The cartesian product of two graphs is a graph denoted by $G \times H$ defined as follows:

$$\mathrm{Ver}(G \times H) = \mathrm{Ver}(G) \times \mathrm{Ver}(H), \quad (gh) \to (g'h') \text{ iff } g \to g' \text{ and } h \to h'.$$

A graph G for which the equality (1) always holds will be called universal.

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3. Let G be a finite graph. $\operatorname{Ver}(G)=\{g_1 \cdots g_n\}$. Let $\{C_1 \cdots C_s\}$ be a fixed ordering of all the different cliques of G. Define $\alpha_i^{(j)}$ as follows:

$$\alpha_i^{(j)} = 1, \qquad g_i \in C_j,$$
$$\qquad = 0, \qquad g_i \notin C_j.$$

Let $P_G=\{(x_1 \cdots x_n) \mid \sum_{i=1}^n \alpha_i^{(j)} x_i \leqq 1,\ x_i \geqq 0,\ 1 \leqq j \leqq s\}$.

THEOREM 2. *A finite graph G is universal if and only if*

$$\max_{x \in P_G} \sum_{i=1}^n x_i = \mu(G), \qquad x = (x_1 \cdots x_n). \tag{2}$$

PROOF. (i) Without loss of generality we may assume that $\{g_1 \cdots g_{\mu(G)}\}=A$ is an independent set of vertices in G. Choose: $x_i=1$, $1 \leqq i \leqq \mu(G)$, $x_i=0$, $i>\mu(G)$. Since no two vertices in A are contained in the same clique it is obvious that for every j we have:

$$\sum_{i=1}^n \alpha_i^{(j)} x_i \leqq 1 \qquad \text{while } \sum_{i=1}^n x_i = \mu(G).$$

Therefore we always have $\max \sum_{i=1}^n x_i \geqq \mu(G)$.

(ii) Suppose G is not universal, i.e. there exists a graph H such that

$$\mu(G \times H) > \mu(G) \cdot \mu(H).$$

(It is obvious that $\mu(G \times H) \geqq \mu(G) \cdot \mu(H)$.) Let $A \subset G \times H$ be a maximal independent set of vertices in $G \times H$ (i.e. card $A=\mu(G \times H)$). Define $A_i=\{h \mid (g_i h) \in A\}$, $(A_i \subset H)$. Since $(g_i h) \rightarrow (g_i h')$ if $h \rightarrow h'$ and A is independent it follows that A_i is an independent set of vertices in H and therefore card $A_i \leqq \mu(H)$. Furthermore if $A_i'=\{(g_i h) \mid h \in A_i\}$ then card $A=\bigcup_{i=1}^n A_i'$ and the union is disjoint. Now choose $x_i=(1/\mu(H))$ card A_i, it is obvious that

$$\sum_{i=1}^n x_i = \frac{1}{\mu(H)} \sum_{i=1}^n \operatorname{card} A_i' = \frac{\mu(G \times H)}{\mu(H)} > \mu(G). \tag{3}$$

Let us show that, for every j, $\sum_{i=1}^n \alpha_i^{(j)} x_i \leqq 1$. If

$$C_j = \{g_{i_1} \cdots g_{i_k}\} \Rightarrow \sum_{i=1}^n \alpha_i^{(j)} x_i = \sum_{l=1}^k x_{i_l}. \tag{4}$$

Since $g_{i_r} \rightarrow g_{i_t}$, $1 \leqq r, t \leqq k$, it follows that $\bigcup_{l=1}^k A_{i_l}$ is an independent set of vertices in H, and the union is disjoint, hence the following holds

$$\mu(H) \sum_{l=1}^k x_i = \sum_{l=1}^k \operatorname{card} A_{i_l} = \operatorname{card}\left\{\bigcup_{l=1}^k A_{i_l}\right\} \leqq \mu(H).$$

Thus, (3) and (4) prove that our condition is necessary.

(iii) To prove the sufficiency of our condition, suppose that

$$\max_{x \in P_G} \sum_{i=1}^{n} x_i > \mu(G).$$

Since P_G is a polytope in the n-dimensional Euclidean space the maximum is obtained; furthermore, since the coefficients of the half spaces determining the polytope are nonnegative integers we may assume without loss of generality that all the components $(x_1 \cdots x_n)$ of the maximizing point which can be chosen to be a vertex of P_G are rational. Let β be the least common multiplier of all the denominators of the x_i's (it is obvious that $\beta < n!$ being the determinant of a matrix of order n with 0's and 1's). Let $y_i = \beta \cdot x_i$, hence $\{y_i\}$ is a set of nonnegative integers satisfying

$$\sum_{i=1}^{n} y_i > \mu(G) \cdot \beta. \tag{5}$$

$$\sum_{i=1}^{n} \alpha_i^{(j)} y_i \leqq \beta, \qquad 1 \leqq j \leqq s. \tag{6}$$

Using (5) and (6) we shall construct a graph H for which the inequality

$$\mu(G \times H) > \mu(H) \cdot \mu(G)$$

will hold. This of course will complete the proof of our theorem.

Let A_i, $1 \leqq i \leqq n$, be a family of n disjoint sets such that card $A_i = y_i$. Let Ver $H = \bigcup_{i=1}^{n} A_i$. Two vertices in H: z, u will be joined by an edge if:

(a) $z = u$ (b) $z \in A_i$, $u \in A_j \rightarrow i \neq j$ and $g_i \neq g_j$.

(Hence any set A_i is independent.)

Let $U = \{u_1 \cdots u_r\}$ be an independent set of vertices in H. We may assume that $U \cap A_i \neq \emptyset$, $1 \leqq i \leqq t$, and $U \cap A_i = \emptyset$, $i > t$. Since U is independent so is $\bigcup_{i=1}^{t} A_i$. It follows from our definition of H that the set $\{g_1 \cdots g_t\}$ is a complete subgraph of G and therefore it is contained in a clique of G. Hence we have: $\sum_{i=1}^{t} x_i \leqq 1$. But this implies: $\sum_{i=1}^{t} y_i \leqq \beta \Rightarrow$ card $\{\bigcup_{i=1}^{t} A_i\} \leqq \beta$. This means that:

$$\mu(H) \leqq \beta. \tag{7}$$

Let $D = \{(g_i h) \mid h \in A_i\}$ $(D \subset G \times H)$. If (g, h), $(g'h') \in D$, then $g \rightarrow g' \Rightarrow h \nrightarrow h'$ and therefore $(gh) \nrightarrow (g'h')$; if $g \nrightarrow g'$ it is obvious that $(gh) \nrightarrow (g'h')$ i.e. D is an independent set of vertices in $G \times H$. Now using (5), (6) and (7) we obtain:

$$\mu(G \times H) \geqq \operatorname{card} D = \operatorname{card}\left\{\bigcup_{i=1}^{n} A_i\right\} = \sum_{i=1}^{n} y_i > \mu(G)\cdot\beta \geqq \mu(G)\cdot\mu(H).$$

This completes the proof of our theorem.

REMARKS. The condition of Theorem 2 can be expressed in terms of discrete linear programming as follows: G is universal if and only if for any set of nonnegative integers x_i satisfying:

$$\sum_{i=1}^{n} \alpha_i^{(j)} x_i \leqq \beta, \qquad 1 \leqq j \leqq s, \qquad \sum_{i=1}^{n} x_i \leqq \mu(G)\cdot\beta$$

for all nonnegative integers β.

Suppose G is not universal, i.e. there exists a β for which $\max \sum_{i=1}^{n} x_i > \mu(G)\cdot\beta$. If $\{g_1 \cdots g_{\mu(G)}\}$ is an independent set of vertices in G, choose $y_i = x_i + 1$ $(1 \leqq i \leqq \mu(G))$, $y_i = x_i$ $(i > \mu(G))$; it is obvious that $\sum_{i=1}^{n} \alpha_i^{(j)} y_i \leqq \beta + 1$ while $\sum_{i=1}^{n} y_i > \mu(G)\cdot(\beta+1)$. This shows that if G is not universal with respect to β it is also not universal with respect to $\beta+1$. Since the number of different graphs (up to isomorphism) with n vertices is finite, it follows that there exists an integer $\beta(n)$ such that G is universal if and only if G is universal with respect to $\beta(n)$. The function $\beta(n)$ is a nondecreasing function of n. The values of $\beta(n)$ for $n \leqq 5$ may be easily computed using Shannon's observation [3] that all graphs with at most 5 vertices are universal except for the pentagon which is not universal with respect to 2. Hence $\beta(n) = 0$, $n \leqq 4$, $\beta(5) = 2$. Using (iii) in the proof of Theorem 2 one can see that $\beta(n) < n!$.

One can use Theorem 2 to estimate the value of $\mu(G \times H)$ as follows: given G and H one can calculate

$$a = \max \sum_{i=1}^{n} x_i$$

subject to

$$\sum_{i=1}^{n} \alpha_i^{(j)} x_i \leqq \mu(H), \quad 1 \leqq j \leqq s_G,$$

where x_i is a nonnegative integer and

$$b = \max \sum_{i=1}^{m} y_i, \quad \sum_{i=1}^{m} \beta_i^{(j)} y_i \leqq \mu(G), \qquad i \leqq j \leqq s_H.$$

($\beta_i^{(j)}$ has the same meaning with respect to H as $\alpha_i^{(j)}$ with respect to G.) It is obvious that: $\mu(G \times H) \leqq \min(a, b)$.

4. In this paragraph we shall show that Shannon's condition is not necessary. Observe first that if G is a graph and σ a preserving function defined on G, and if $A \subset G$ is an independent set of vertices, then

$\sigma(A)$ is independent and card$\{\sigma(A)\}$ = card A; therefore we always have $\mu(\sigma(G)) = \mu(G)$. Since $\sigma^{-1}(g)$ is a complete subgraph of G it follows that G is covered by card$\{\text{Ver } \sigma(G)\}$ complete subgraphs, therefore a necessary condition for the existence of a preserving function σ such that $\sigma(G)$ is independent is that G is covered by $\mu(G)$ complete subgraphs.

Let G_1 and G_2 be two disjoint pentagons and G_3 a set of 5 vertices no one of which belongs to G_1 or G_2. Adjoin by an edge each vertex of G_3 to all the vertices of G_1 and G_2. Let H be the graph defined by these relations, hence we have:

$$\text{card}\{\text{Ver } H\} = 15, \quad \mu(H) = 5 \qquad (G_3 \text{ is independent}).$$

Since a pentagon cannot be covered by less than 3 complete subgraphs, it is obvious that H cannot be covered by less than 6 complete subgraphs. Thus we have shown that Shannon's condition cannot hold for H.

To show that H is universal observe that all the cliques of H are triangles, every vertex of H is contained in exactly 10 different cliques and the number of different cliques is 50, therefore the following holds:

$$1 \leqq j \leqq 50, \qquad \sum_{i=1}^{15} \alpha_i^{(j)} x_i \leqq 1 \Rightarrow \sum_{j=1}^{50} \sum_{i=1}^{15} \alpha_i^{(j)} x_i \leqq 50,$$

but:

$$\sum_{j=1}^{50} \sum_{i=1}^{15} \alpha_i^{(j)} x_i = 10 \sum_{i=1}^{15} x_i \leqq 50 \Rightarrow \max \sum_{i=1}^{15} x_i \leqq 5 = \mu(H)$$

and, by Theorem 2, H is universal. Q.E.D.

References

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