A GENERALIZED APPROXIMATION THEOREM FOR DEDEKIND DOMAINS

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It is well known that a Dedekind domain $A$ with a finite number of prime ideals is a principal ideal domain. A reasonable generalization of this result would be: If $A$ is a Dedekind domain and $S$ is the set of prime ideals of $A$, then $\text{card } S < \text{card } A$ implies that $A$ is a principal ideal domain.

In fact, this latter statement is false; see [1]. But it is true that if $(\text{card } S)^{\aleph_0} < \text{card } A$, then $A$ is a principal ideal domain. A proof is given in the present article of a slight generalization (analogous to the weak approximation theorem) of this result.

Before proceeding to this result, we give a proposition that displays a large class of examples for which the stronger assertion of the first paragraph is valid. We will use the phrase “Let $A$, $S$ be a Dedekind domain” rather than “Let $A$ be a Dedekind domain, and let $S$ be the set of prime ideals of $A$” for the balance of the article. Also, if $P$ is a prime ideal of $A$, then $v_P$ will denote the normed valuation going with the prime ideal $P$.

**Proposition.** Let $A$, $S$ be a Dedekind domain and suppose that $A$ contains a field $F$ such that $\text{card } F = \text{card } A$. Suppose that $\text{card } S < \text{card } A$. Then $A$ is a principal ideal domain.

**Proof.** Let $P$ be in $S$; choose $\pi$ and $\sigma$ in $A$ such that $v_P(\pi) = 1$, $v_P(\sigma) = 2$ and $P = (\pi, \sigma)$. Consider the set of elements $\pi + f\sigma$ for $f$ in $F$. For all $f$, we have $v_P(\pi + f\sigma) = 1$. If $P$ is not principal, then for each $f$, there must be a $Q \neq P$ in $S$ such that $\pi + f\sigma$ is in $Q$. Since card $F > \text{card } S$, there will be an $f$ and $f'$ in $F$ such that $\pi + f\sigma$ and $\pi + f'\sigma$ are in the same prime ideal $Q \neq P$ of $S$. But then $(f - f')\sigma$ is in $Q$, so $\sigma$ is $Q$, forcing also $\pi$ in $Q$. This implies that $P' = (\pi, \sigma)$ is contained in $Q$, which is a contradiction.

**Remark.** If $T$ is a subset of $S$, $P$ is a prime ideal in $T$, and there is an element $a_P$ of $A$ such that $v_Q(a_P) = \delta_{P,Q}$ for all $Q$ in $T$, then we will say that $P$ is principal with respect to $T$.

**Theorem.** Let $A$, $S$ be a Dedekind domain and let $T$ be a subset of $S$ such that not every prime ideal of $T$ is principal with respect to $T$. Then $\text{card } A \leq (\text{card } T)^{\aleph_0}$.

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Proof. If \( T \) is a finite set, the weak approximation theorem handles the situation. If any prime ideal \( P \) of \( S \) is such that \( A/P \) is finite, then imbedding \( A \) in its \( P \)-adic completion shows that \( \text{card } A \leq (\text{card } A/P)^{\aleph_0} \leq (\text{card } T)^{\aleph_0} \); so we may assume that \( A/P \) is infinite for any \( P \) in \( S \).

We first show that there is a subset \( S' \) of \( S \) such that \( T \) is contained in \( S' \), \( \text{card } T = \text{card } S' \), and \( S' \) contains an infinite number of prime ideals which are not principal with respect to \( S' \). If \( T \) will not work, let \( P_1, \ldots, P_k \) be the prime ideals of \( T \) which are not principal with respect to \( T \). For each \( Q \) in \( T - \{ P_1, \ldots, P_k \} \) let \( a_Q \) be a generator of \( Q \) with respect to \( T \). Let \( W \) be the set of prime ideals of \( S - T \) which contain an \( a_Q \) for some \( Q \) in \( T - \{ P_1, \ldots, P_k \} \). Set \( T' = T \cup W \). \( P_1, \ldots, P_k \) are certainly not principal with respect to \( T' \); further, not all prime ideals \( R \) in \( W \) can be principal with respect to \( T' \). For suppose they are, and let \( a_R \) \((R \text{ in } W) \) be a generator for \( R \) with respect to \( T' \). Choose \( x \) in \( A \) such that \( v_{P_1}(x) = 1, v_{P_i}(x) = 0 \) for \( i = 2, \ldots, k \). Multiplying \( x \) by appropriate negative powers of the \( a_Q \) \((Q \text{ in } T - \{ P_1, \ldots, P_k \}) \) yields an element \( y \) in the quotient field of \( A \) such that \( v_{P_1}(y) = 1, v_{P_i}(y) = 0 \) for \( i = 2, \ldots, k \), and \( v_Q(y) = 0 \) for \( Q \) in \( T - \{ P_1, \ldots, P_k \} \). But \( y \) will have negative values only for certain of the \( v_R \) \((R \text{ in } W) \). Multiplying by appropriate positive powers of the \( a_R \) \((R \text{ in } W) \) produces an element which generates \( P_1 \) with respect to \( T' \) (hence with respect to \( T \)) and gives a contradiction.

Inductively, set \( T_1 = T' \) and \( T_n = (T_{n-1})' \). Then \( U_1 T_n \) works.

We will now assume that \( T \) contains an infinite sequence of primes \( P_1, P_2, \ldots \), which are not principal with respect to \( T \). We will let \( v_i \) denote the valuation going with \( P_i \). Choose \( \pi_1 \) and \( \sigma_1 \) such that \( v_1(\pi_1) = 1, v_1(\sigma_1) = 2 \) and \( P_1 = (\pi_1, \sigma_1) \). Delete from the list the \( P_i \) \((i > 1) \) which contain \( \sigma_1 \) and renumber the remaining primes in their original order. Choose \( \pi_2 \) and \( \sigma_2 \) such that \( v_2(\pi_2) = 1, v_2(\sigma_2) = 2 \), \( P_2 = (\pi_2, \sigma_2) \) and \( \pi_2/\sigma_2 \not\equiv \pi_1/\sigma_1 \). Delete from the list the \( P_i \) \((i > 2) \) which contain \( \sigma_1 \) or \( \sigma_2 \) or for which \( v_i(\pi_2/\sigma_2 - \pi_1/\sigma_1) > 0 \). Inductively, choose \( \pi_j \) and \( \sigma_j \) such that \( v_j(\pi_j) = 1, v_j(\sigma_j) = 2 \), \( P_j = (\pi_j, \sigma_j) \) and also subject to the condition: if \( k < j \), then \( \pi_j/\sigma_j \not\equiv \pi_k/\sigma_k \) modulo any of the prime ideals \( Q \) of \( S \) for which \( v_Q(\pi_m/\sigma_m - \pi_n/\sigma_n) \) is positive with \( m, n < j \) (this of course provided \( v_Q(\pi_k/\sigma_k) \geq 0 \)). This is possible since \( A/P \) is infinite for all \( P \) in \( S \) and since \( v_i \) is not positive at \( \pi_m/\sigma_m - \pi_n/\sigma_n \) with \( m, n < j \). Then delete the \( P_i \) \((i > j) \) which contain any of \( \sigma_1, \ldots, \sigma_j \) or for which \( v_i(\pi_m/\sigma_m - \pi_n/\sigma_n) > 0 \) with \( m, n \leq j \) and renumber.

Let \( a \) be an element of \( A \), and consider the set \( \{ \pi_i + a\sigma_i \} \). For each \( i \), \( v_i(\pi_i + a\sigma_i) = 1 \), and \( P_i \) is not principal with respect to \( T \), so there is a prime ideal \( Q_i \) \((\not\equiv P_i) \) in \( T \) such that \( \pi_i + a\sigma_i \) is in \( Q_i \). Making
some choice for each $i$, let $f_a: i \rightarrow Q_i$ be the map induced by $a$. The image of $f_a$ is infinite. If not, then there is a finite subset $R_1, \ldots, R_m$ of $T$ such that $\pi_i + a\sigma_i$ is contained in one of these for each $i$. Suppose that for an infinite number of $i$, $\pi_i + a\sigma_i$ is in $R_1$. If $R_1$ is a $P_n$ for some $n$, choose $p, q, r$ such that $p > q > r > n$ with $\pi_p + a\sigma_p, \pi_q + a\sigma_q, \pi_r + a\sigma_r$ all in $R_1$ (if $R_1$ is not in the set $\{P_i\}$, simply choose $p > q > r$). If $\sigma_j$ is in $R_1$ for $j = p, q, r$, then $\pi_j$ is also in $R_1$ forcing $P_j$ to be $R_1$ and giving a contradiction. We then have $\pi_j/\sigma_j = -a(R_1)$ for $j = p, q, r$ and so we get $\pi_p/\sigma_p \equiv \pi_q/\sigma_q$ modulo $R_1$, but $v_{R_1}$ has positive value at $\pi_q/\sigma_q - \pi_r/\sigma_r$ which contradicts the construction for the $\pi_i$ and $\sigma_i$.

If now $f_a = f_b$, we get that $\pi_i + a\sigma_i$ and $\pi_i + b\sigma_i$ are in the same ideal $Q_i$ for all $i$. This yields $(a - b)\sigma_i$ in $Q_i$ for all $i$. We cannot have $\sigma_i$ in $Q_i$ (else $P_i = Q_i$) so $a - b$ is in $Q_i$ for all $i$. Since the set $\{Q_i\}$ is infinite, $a = b$.

Letting $N$ denote the natural numbers, we get that each $a$ in $A$ induces a map $f_a: N \rightarrow T$. Since $a \rightarrow f_a$ is one-to-one, we have $\text{card } A \leq (\text{card } T)^{\aleph_0}$.

**Corollary 1 (Generalized Approximation Theorem).** Let $A$, $S$ be a Dedekind domain, and let $T$ be a subset of $S$ such that $(\text{card } T)^{\aleph_0} < \text{card } A$. Given a set of nonnegative integers $\{n_p\}$ for $P$ in $T$ which are almost all zero, there is an element $x$ in $A$ such that $v_p(x) = n_p$ for all $P$ in $T$.

**Corollary 2.** If $A$, $S$ is a Dedekind domain and $(\text{card } S)^{\aleph_0} < \text{card } A$, then $A$ is a principal ideal domain.

**References**


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