A GENERALIZED APPROXIMATION THEOREM FOR DEDEKIND DOMAINS

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It is well known that a Dedekind domain A with a finite number of prime ideals is a principal ideal domain. A reasonable generalization of this result would be: If A is a Dedekind domain and S is the set of prime ideals of A, then card $S <$ card A implies that A is a principal ideal domain.

In fact, this latter statement is false; see [1]. But it is true that if $(\text{card } S)^{\aleph_0} < \text{card } A$, then A is a principal ideal domain. A proof is given in the present article of a slight generalization (analogous to the weak approximation theorem) of this result.

Before proceeding to this result, we give a proposition that displays a large class of examples for which the stronger assertion of the first paragraph is valid. We will use the phrase "Let A, S be a Dedekind domain" rather than "Let A be a Dedekind domain, and let S be the set of prime ideals of A" for the balance of the article. Also, if P is a prime ideal of A, then v_p will denote the normed valuation going with the prime ideal P.

PROPOSITION. *Let A, S be a Dedekind domain and suppose that A contains a field F such that* card $F =$ card A. *Suppose that* card $S <$ card A. *Then A is a principal ideal domain.*

PROOF. Let P be in S; choose π and σ in A such that $v_p(\pi) = 1$, $v_p(\sigma) = 2$ and $P = (\pi, \sigma)$. Consider the set of elements $\pi + f\sigma$ for f in F. For all f, we have $v_p(\pi + f\sigma) = 1$. If P is not principal, then for each f, there must be a $Q \neq P$ in S such that $\pi + f\sigma$ is in Q. Since card F $>$ card S, there will be an f and f' in F such that $\pi + f\sigma$ and $\pi + f'\sigma$ are in the same prime ideal $Q \neq P$ of S. But then $(f - f')\sigma$ is in Q, so σ is Q, forcing also π in Q. This implies that $P = (\pi, \sigma)$ is contained in Q, which is a contradiction.

REMARK. If T is a subset of S, P is a prime ideal in T, and there is an element a_P of A such that $v_Q(a_P) = \delta_{P,Q}$ for all Q in T, then we will say that P is principal with respect to T.

THEOREM. *Let A, S be a Dedekind domain and let T be a subset of S such that not every prime ideal of T is principal with respect to T. Then* card $A \leqq (\text{card } T)^{\aleph_0}$.

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PROOF. If T is a finite set, the weak approximation theorem handles the situation. If any prime ideal P of S is such that A/P is finite, then imbedding A in its P-adic completion shows that card $A \leqq (\text{card } A/P)^{\aleph_0} \leqq (\text{card } T)^{\aleph_0}$; so we may assume that A/P is infinite for any P in S.

We first show that there is a subset S' of S such that T is contained in S', card $T=$card S', and S' contains an infinite number of prime ideals which are not principal with respect to S'. If T will not work, let $P_1, \cdots, P_k$ be the prime ideals of T which are not principal with respect to T. For each Q in $T-\{P_1, \cdots, P_k\}$ let a_Q be a generator of Q with respect to T. Let W be the set of prime ideals of $S-T$ which contain an a_Q for some Q in $T-\{P_1, \cdots, P_k\}$. Set $T'=T\cup W$. $P_1, \cdots, P_k$ are certainly not principal with respect to T'; further, not all prime ideals R in W can be principal with respect to T'. For suppose they are, and let a_R (R in W) be a generator for R with respect to T'. Choose x in A such that $v_{p_1}(x)=1$, $v_{p_i}(x)=0$ for $i=2, \cdots, k$. Multiplying x by appropriate negative powers of the a_Q (Q in $T-\{P_1, \cdots, P_k\}$) yields an element y in the quotient field of A such that $v_{p_1}(y)=1$, $v_{p_1}(y)=0$ for $i=2, \cdots, k$, and $v_Q(y)=0$ for Q in $T-\{P_1, \cdots, P_k\}$. But y will have negative values only for certain of the v_R (R in W). Multiplying by appropriate positive powers of the a_R (R in W) produces an element which generates P_1 with respect to T' (hence with respect to T) and gives a contradiction.

Inductively, set $T_1=T'$ and $T_n=(T_{n-1})'$. Then $\bigcup_1^\infty T_n$ works.

We will now assume that T contains an infinite sequence of primes $P_1, P_2, \cdots$, which are not principal with respect to T. We will let v_i denote the valuation going with P_i. Choose π_1 and σ_1 such that $v_1(\pi_1)=1$, $v_1(\sigma_1)=2$ and $P_1=(\pi_1, \sigma_1)$. Delete from the list the P_i $(i>1)$ which contain σ_1 and renumber the remaining primes in their original order. Choose π_2 and σ_2 such that $v_2(\pi_2)=1$, $v_2(\sigma_2)=2$, $P_2=(\pi_2, \sigma_2)$ and $\pi_2/\sigma_2\neq\pi_1/\sigma_1$. Delete from the list the P_i $(i>2)$ which contain σ_1 or σ_2 or for which $v_i(\pi_2/\sigma_2-\pi_1/\sigma_1)>0$. Inductively, choose π_j and σ_j such that $v_j(\pi_j)=1$, $v_j(\sigma_j)=2$, $P_j=(\pi_j, \sigma_j)$ and also subject to the condition: if $k<j$, then $\pi_j/\sigma_j\not\equiv\pi_k/\sigma_k$ modulo any of the prime ideals Q of S for which $v_Q(\pi_m/\sigma_m-\pi_n/\sigma_n)$ is positive with $m, n<j$ (this of course provided $v_Q(\pi_k/\sigma_k)\geqq 0$). This is possible since A/P is infinite for all P in S and since v_j is not positive at $\pi_m/\sigma_m-\pi_n/\sigma_n$ with $m, n<j$. Then delete the P_i $(i>j)$ which contain any of $\sigma_1, \cdots, \sigma_j$ or for which $v_i(\pi_m/\sigma_m-\pi_n/\sigma_n)>0$ with $m, n\leqq j$ and renumber.

Let a be an element of A, and consider the set $\{\pi_i+a\sigma_i\}$. For each i, $v_i(\pi_i+a\sigma_i)=1$, and P_i is not principal with respect to T, so there is a prime ideal Q_i $(\neq P_i)$ in T such that $\pi_i+a\sigma_i$ is in Q_i. Making some choice for each i, let f_a: $i \to Q_i$ be the map induced by a. The image of f_a is infinite. If not, then there is a finite subset $R_1, \cdots, R_m$ of T such that $\pi_i + a\sigma_i$ is contained in one of these for each i. Suppose that for an infinite number of i, $\pi_i + a\sigma_i$ is in R_1. If R_1 is a P_n for some n, choose p, q, r such that $p > q > r > n$ with $\pi_p + a\sigma_p$, $\pi_q + a\sigma_q$, $\pi_r + a\sigma_r$ all in R_1 (if R_1 is not in the set $\{P_i\}$, simply choose $p > q > r$). If σ_j is in R_1 for $j = p$, q, or r, then π_j is also in R_1 forcing P_j to be R_1 and giving a contradiction. We then have $\pi_j/\sigma_j \equiv -a(R_1)$ for $j = p$, q, r and so we get $\pi_p/\sigma_p \equiv \pi_q/\sigma_q$ modulo R_1, but v_{R_1} has positive value at $\pi_q/\sigma_q - \pi_r/\sigma_r$ which contradicts the construction for the π_i and σ_i.

If now $f_a = f_b$, we get that $\pi_i + a\sigma_i$ and $\pi_i + b\sigma_i$ are in the same ideal Q_i for all i. This yields $(a-b)\sigma_i$ in Q_i for all i. We cannot have σ_i in Q_i (else $P_i = Q_i$) so $a-b$ is in Q_i for all i. Since the set $\{Q_i\}$ is infinite, $a = b$.

Letting N denote the natural numbers, we get that each a in A induces a map f_a: $N \to T$. Since $a \to f_a$ is one-to-one, we have card $A \leqq (\text{card } T)^{\aleph_0}$.

Corollary 1 (Generalized Approximation Theorem). *Let A, S be a Dedekind domain, and let T be a subset of S such that* $(\text{card } T)^{\aleph_0} < \text{card } A$. *Given a set of nonnegative integers $\{n_p\}$ for P in T which are almost all zero, there is an element x in A such that $v_p(x) = n_p$ for all P in T.*

Corollary 2. *If A, S is a Dedekind domain and* $(\text{card } S)^{\aleph_0} < \text{card } A$, *then A is a principal ideal domain.*

References

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