ON p-MIXED ABELIAN GROUPS[1]

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1. **Introduction.** In this paper we investigate the structure of a class of mixed Abelian groups. The word group shall always designate an Abelian group. A mixed group is one which generally contains elements of both finite and infinite order. Such a group, G, contains a unique maximal torsion subgroup, tG; when tG is p-primary we call G a p-mixed group.

The structure theorem we seek is one in which we can write the mixed group as a direct sum of two subgroups, one of which tends to be "tied into" the torsion subgroup more securely; or alternately to find torsion-free direct summands of the group.

$G = \sum G_\alpha$, $\alpha \in I$, means that G is a direct sum of the G_α where α ranges over the index set I. $\operatorname{Ext}(B, A)$ is the group of extensions of A by B. Finally $g \in G$ is said to have p-height n (where p is a prime) if the equation $p^n x = g$ is solvable in G but $p^{n+1}x = g$ is not. If $p^n x = g$ is solvable for all nonnegative integers n then g is said to have infinite p-height. The p-height of an element g in G is designated by $H_p^G(g)$. We let $p^\omega G$ be the set of all elements in G which have infinite p-height in G.

2. **Preliminary lemmas.**

LEMMA 1. *Let J be a torsion-free group and p be a prime. Then $p^\omega J$ is a pure subgroup of J.*

PROOF. Clearly $p^\omega J$ is a subgroup of J. To show that it is a pure subgroup of J we must show that if $nx = g$ is solvable in J for $g \in p^\omega J$ and n some nonnegative integer then $x \in p^\omega J$. Now $nx = g$ gives

$$\infty = H_p^J(g) = H_p^J(nx) = r + H_p^J(x),$$

where p^r is the highest power of p dividing n. Hence $H_p^J(x) = \infty$ which puts x in $p^\omega J$.

With G as our p-mixed group we let $\hat{G}$ denote the factor group G/tG and ν the natural quotient map $\nu\colon G \to \hat{G}$. Lemma 1 says that $p^\omega \hat{G}$ is a pure subgroup of $\hat{G}$. Now let $S = \nu^{-1}(p^\omega \hat{G}) = \{g \in G \mid \nu(g) \in p^\omega \hat{G}\}$. Then $tG \leq S$; also S is a pure subgroup for if $ng \in S$ for $g \in G$ and n a nonnegative integer then

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$$\nu(ng) = n\nu(g) \in p^\omega \hat{G}.$$

Since $p^\omega \hat{G}$ is a pure subgroup of $\hat{G}$ we have $\nu(g) \in p^\omega \hat{G}$ and hence $g \in S$.

One would suspect that among all the subgroups of G which contain tG it is precisely S that the elements of tG are "tied into" most securely. For though the elements of G not in S may experience a change in p-height in passing from G to $\hat{G}$ it is the elements of S that may change most radically for such an element may have finite p-height in G but has infinite p-height in $\hat{G}$. As we shall see this suspicion is confirmed at least when the factor group $\hat{G}/p^\omega \hat{G}$ is a completely decomposable group. (A completely decomposable group is a torsion-free group which is a direct sum of groups of rank one. Note that $\hat{G}/p^\omega \hat{G}$ is a torsion-free group since $p^\omega \hat{G}$ is a pure subgroup of $\hat{G}$ as we have seen in Lemma 1.)

LEMMA 2. *Let J be a torsion-free group and p a prime. Then* $H_p^J(g) = H_p^{J/p^\omega J}(g + p^\omega J)$ *for* $g \in J$.

PROOF. Since $J/p^\omega J$ is a homomorphic image of J we have, for any $g \in J$, $H_p^{J/p^\omega J}(g + p^\omega J) \geqq H_p^J(g)$. So we need only show that if the coset $g + p^\omega J$ is divisible by p^n in $J/p^\omega J$ then g is divisible by p^n in J.

Let $p^n(g_n + p^\omega J) = g + p^\omega J$ where $g_n \in J$. It follows that $p^n g_n - g = y \in p^\omega J$. Hence $H_p^J(y) = \infty$ so there is a $y_n \in J$ such that $p^n y_n = y$. Then $p^n(g_n - y_n) = g$.

Now all the elements of J which have infinite p-height are in $p^\omega J$. So if $g \in J$, $g \notin p^\omega J$ we have just shown that the p-height of g does not change when we pass to $J/p^\omega J$. This gives us

COROLLARY. $p^\omega(J/p^\omega J) = 0$.

In our own special setting where G is a p-mixed group the corollary tells us that $\hat{G}/p^\omega \hat{G}$ has no elements of infinite p-height.

3. **The Main Theorem.**

Lyapin (cf. [2, Theorem 46.5]) has given a necessary and sufficient condition that $p^\omega \hat{G}$ be a direct summand of G when one assumes that $\hat{G}/p^\omega \hat{G}$ is completely decomposable. In this setting we prove

THEOREM. *Let G be a p-mixed group. Suppose $\hat{G}/p^\omega \hat{G}$ is completely decomposable. Then S is a direct summand of G (with complementary summand isomorphic to the torsion-free group $\hat{G}/p^\omega \hat{G}$) if and only if $p^\omega \hat{G}$ is a direct summand of $\hat{G}$.*

PROOF. Suppose $p^\omega \hat{G}$ is a direct summand of $\hat{G}$. Then we can write $\hat{G} = p^\omega \hat{G} \oplus \hat{K}$ where $\hat{K}$ is a subgroup of $\hat{G}$ and is completely decomposable since $\hat{K} \cong \hat{G}/p^\omega \hat{G}$. So $\hat{K} = \sum J_\alpha$, $\alpha \in I$, where each J_α is a group of rank one. If we denote the type (cf. [2, Chapter 7]) of J_α by $\tau(J_\alpha) = (s_1^\alpha, s_2^\alpha, \cdots, s_n^\alpha, \cdots)$ where the s_i are nonnegative integers or ∞, it follows that $s_j^\alpha < \infty$ for each $\alpha \in I$ where s_j^α corresponds to the prime p. This is so because the corollary to Lemma 2 tells us that $\hat{K}$ has no elements of infinite p-height so certainly no J_α can have such elements.

Now for any group T, $\operatorname{Ext}(\sum J_\alpha, T) \cong \prod \operatorname{Ext}(J_\alpha, T)$ where $\prod$ denotes the unrestricted direct sum (taken over the index set I). In particular if T is any p-primary torsion-group we have, by Baer's theorem (cf. [1]), $\operatorname{Ext}(J_\alpha, T) = 0$ for every $\alpha \in I$ since $p^\omega J_\alpha = 0$. Hence $\operatorname{Ext}(\sum J_\alpha, T) = 0$. But $\hat{K} \cong \sum J_\alpha$ and tG is a p-primary torsion group so $\operatorname{Ext}(\hat{K}, tG) = 0$.

Now let $K' = \nu^{-1}(\hat{K})$. Then K' is an extension of tG by $\hat{K}$. But we have just seen that all such extensions split. So we can find a torsion-free subgroup $K \leqq K' \leqq G$ such that $K' = K \oplus tG$. Clearly $G = S + K'$, recalling that $S = \nu^{-1}(p^\omega \hat{G})$, though the sum is not direct since $K' \cap S = tG$. On the other hand $G = S + K' = S + (K + tG) = S + K$ since $tG \leqq S$. Now this sum is direct, i.e. $G = S \oplus K$, since $S \cap K = 0$. Hence S is a direct summand of G.

The converse, namely $G \cong S \oplus \hat{G}/p^\omega \hat{G}$ implies $\hat{G} \cong p^\omega \hat{G} \oplus \hat{G}/p^\omega \hat{G}$, is clear.

COROLLARY. *Suppose $\hat{G}$ is completely decomposable. Then S is a direct summand of G.*

PROOF. Since $\hat{G}$ is completely decomposable we can write $\hat{G} = \sum G_\alpha$, $\alpha \in I$. Now $p^\omega \hat{G} = \sum G_\beta$, $\beta \in I_0 \leqq I$, where $\tau(G_\beta)$ is infinite at the prime p and so $p^\omega \hat{G}$ is a direct summand of $\hat{G}$. Now $\hat{G}/p^\omega \hat{G}$ is isomorphic to the complementary summand of $p^\omega \hat{G}$ and so can be considered a direct summand of the completely decomposable group $\hat{G}$. So, by [4] and [3] $\hat{G}/p^\omega \hat{G}$ is a completely decomposable group. Thus we can apply the theorem to get that S is a direct summand of G.

REMARK. The theorem could have been stated in a more general setting by demanding that $\hat{G}/p^\omega \hat{G}$ be "not almost p-infinite" (cf. [1]) rather than being completely decomposable.

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