FINITE p-SOLVABLE LINEAR GROUPS WITH A CYCLIC SYLOW p-SUBGROUP

D. L. WINTER

In [3] N. Itô proved that if p is a prime and if G is a finite p-solvable linear group over the complex number field of degree less than $p-1$, then G has a normal abelian Sylow p-subgroup. In this paper the same type of proof (see also [1, p. 143]) will be used for the following result.

THEOREM. *Let p be a prime and let G be a finite p-solvable group which contains a cyclic Sylow p-subgroup of order p^a. If G has a faithful representation over the complex number field of degree less than $p^{a-t}(p-1)$ where t is an integer such that $1 \leqq t \leqq a$, then G has a normal subgroup of order p^t.*

Let G be a counterexample to the theorem of minimal order. Let P be a fixed Sylow p-subgroup of G of order p^a, P_0 the unique subgroup of P of order p^t. Clearly, G is not a p-group. Let $|G|$ have at least three distinct prime divisors. Since G is p-solvable, G contains a (p, q)-Hall subgroup $S(p, q)$ for any prime q which is distinct from p [4, page 196]. Since $G \neq S(p, q)$, $P_0 \lhd S(p, q)$ by the induction hypothesis. Since this is true for any q which is distinct from p, $P_0 \lhd G$. Hence we may assume that $|G| = p^a q^b$ for some prime $q \neq p$.

From now on χ will denote a fixed faithful character of G (i.e., a character of a faithful representation of G) of minimal degree. A contradiction will be obtained after a series of short steps.

(1) *χ is irreducible.*

PROOF. Suppose χ is reducible. Let χ_1 be a nonlinear irreducible constituent of χ and let K be its kernel. Then $1 \neq K \neq G$ and since χ_1 is nonlinear, $PK \neq G$.

It can now be seen that $P_0K \lhd G$. If $P_0 \leqq K$, this is clear. Hence assume that $|K| = p^c q^d$, $0 \leqq c < t$, $0 \leqq d \leqq b$. Since χ_1 is a faithful character of G/K of degree less than $p^{a-t}(p-1) = p^{(a-c)-(t-c)}(p-1)$, the induction hypothesis implies that G/K contains a normal subgroup of order p^{t-c}. Let $P_1 \leqq P$ be such that P_1K/K is this group. Then $P_1K \lhd G$ and $|P_1| = p^t$ and hence $P_1 = P_0$.

The induction hypothesis implies that $P_0 \lhd PK$. Therefore $P_0 \lhd P_0K$ whence $P_0 \lhd G$. This contradiction proves (1).

Let Q be a Sylow q-subgroup of G.

(2) $Q \lhd G$.

Received by the editors July 12, 1966.

PROOF. Let Q_0 be the maximal normal q-subgroup of G. Then G/Q_0 contains no normal q-subgroup. Let $P_1 \leqq P$ be such that P_1Q_0/Q_0 is the maximal normal p-subgroup of G/Q_0. By Lemma 1.2.3 of [2], P_1Q_0/Q_0 contains its centralizer in G/Q_0. Hence, $P \leqq P_1Q_0$ and so $P = P_1$ and $PQ_0 \lhd G$. The induction hypothesis forces $PQ_0 = G$. Hence, Q_0 is a Sylow q-subgroup of G as was to be shown.

(3) *G contains no normal p-subgroup.*

Suppose on the contrary that G contains a proper normal p-subgroup U. Then $P \leqq C(U) \lhd G$. If $C(U) \neq G$, then $P_0 \lhd C(U)$ and so $P_0 \lhd G$, a contradiction. Therefore $C(U) = G$ and $p \mid |Z(G)|$ where $Z(G)$ is the center of G. Let $P_1 = P \cap Z(G)$. Then $\chi_{|P_1} = \chi(1)\mu$ where μ is a linear character of P_1. Let λ be a linear character of G/Q such that $\lambda_{|P_1} = \bar{\mu}$. Then $\lambda\chi$ is a faithful character of G/P_1. The induction hypothesis yields that (since $P_1 \neq P_0$) $P_0/P_1 \lhd G/P_1$ and hence $P_0 \lhd G$, a contradiction.

(4) *$\chi_{|Q}$ is irreducible. Hence Q is nonabelian.*

Suppose $\chi_{|Q}$ is reducible. Let P_1 be the maximal subgroup of P such that $\chi_{|P_1Q}$ is reducible. Since χ is an irreducible character of G, $P_1 \neq P$. Let P_2 be the unique subgroup of P such that $|P_2 : P_1| = p$. By maximality of P_1, $\chi_{|P_2Q}$ is irreducible. By [1, pp. 54–55], $\chi_{|P_1Q}$ is a sum of p distinct conjugate characters and if θ is one of these, the inertia group of θ in P_2Q is P_1Q. But this implies that the inertia group of θ in G is P_1Q. Therefore the induced character θ^* is irreducible and $\chi = \theta^*$. Hence $\chi(1) = |G : P_1Q|\theta(1) = p\theta(1)$ and so $|G : P_1Q| = p$. Now $\theta(1) = \chi(1)/p < p^{(a-1)-t}(p-1)$. Thus $\chi_{|P_1Q}$ is a faithful character of P_1Q all of whose irreducible constituents have degree less than $p^{(a-1)-t}(p-1)$. Let $K_1, \cdots, K_p$ be the kernels of these constituents. If $t \leqq a-1$, the induction hypothesis implies that $K_iP_0 \lhd P_1Q$ for all i. If this is the case, then $P_0 = K_1P_0 \cap K_2P_0 \cap \cdots \cap K_pP_0 \lhd P_1Q$ and so $P_0 \lhd G$. Since this cannot occur, $t > a-1$ or $t = a$, contradicting [3]. This proves (4).

(5) *If Q_0 is a proper subgroup of Q normal in G, then $Q_0 \leqq Z(G)$.*

PROOF. Since $Q_0 \lhd G$, PQ_0 is a group and $PQ_0 \neq G$ since $Q_0 \neq Q$. By induction $P_0 \lhd PQ_0$. This implies that $P_0Q_0 = P_0 \times Q_0$ and $P_0 \leqq C(Q_0) \lhd G$. Suppose $C(Q_0) \neq G$. Then $\chi_{|C(Q_0)}$ must be reducible for otherwise Schur's Lemma would imply $Q_0 \leqq Z(G)$, each element of Q_0 commuting with the irreducible system $C(Q_0)$. P is not contained in $C(Q_0)$ since otherwise induction yields $P_0 \lhd C(Q_0)$ and so $P_0 \lhd G$.

Suppose $PC(Q_0) \neq G$. Then $P_0 \lhd PC(Q_0)$ and hence $P_0 \lhd C(Q_0)$ which is not the case. Therefore, $PC(Q_0) = G$ and $|G : C(Q_0)|$ is a power of p. This implies that $\chi_{|Q}$ is reducible, contradicting (4). (5) is now proved.

As G has no normal subgroup of index q, neither does G/Q'. Now G/Q' has an abelian q-Sylow subgroup. Suppose there exists $Q_0 \lhd G$ with $Q' < Q_0 < Q$. By (5) $Q_0 \leqq Z(G)$ and Q_0/Q' is a subgroup of the center of G/Q'. Therefore [**5**, page 173] G/Q' contains a normal subgroup of index q, which is a contradiction. This proves

(6) *Q/Q' is a minimal normal subgroup of G/Q'.*

(7) *Let $N(P)$ and $C(P)$ be, respectively, the normalizer and the centralizer of P in G. Then $N(P) = C(P) = P \times Q'$.*

PROOF. Let $Q_1 = N(P) \cap Q$. Then $N(P) = P \times Q_1 \leqq C(P)$. Hence, $N(P) = C(P)$. $Q' \leqq Q_1$ by (5) and so $Q_1 \lhd Q$ since Q/Q' is abelian. Therefore, $Q_1 \lhd G$. By (6), $Q_1 = Q'$, proving (7).

By (5), $Q' \leqq Z(Q)$ and by (6), $Q' = Z(Q)$. Since $Q' \leqq Z(G)$, Q' is cyclic, χ being irreducible and faithful. By (6), Q/Q' has exponent q. Hence by [**1**, p. 142],

(8) *$|Z(Q)| = q$, $|Q : Q'| = q^{2n}$ for some integer n and every nonlinear character of Q has degree q^n.*

Since the cyclic group P is faithful on the chief factor Q/Q', it follows that p^a divides $q^{2n} - 1$, and so $q^n \equiv \pm 1 \bmod p^a$. This implies that

$$p^a \sim q^n + 1 = \chi(1) + 1 < p^{a-t}(p - 1) + 1 = p^{a-t+1} - p^{a-t} + 1$$
$$< p^{a-t+1} \sim p^a,$$

a final contradiction.

REFERENCES

1. W Feit, *Characters of finite groups*, Lecture notes, Yale Univ., New Haven, Conn., 1965.

2. P. Hall and G. Higman, *The p-length of a p-soluble group and reduction theorems for Burnside's problem*, Proc. London Math. Soc. **6** (1956), 1–42.

3. N. Itô, *On a theorem of H. F. Blichfeldt*, Nagoya Math. J. **5** (1954), 75–77.

4. A. G. Kurosh, *The theory of groups*, Vol. II, Chelsea, New York, 1956.

5. H. Zassenhaus, *The theory of groups*, Chelsea, New York, 1958.

MICHIGAN STATE UNIVERSITY