TOPOLOGICAL TRANSFORMATION GROUPS WITH A FIXED END POINT

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1. Let (X, T, π) be a topological transformation group, where X is a nontrivial Hausdorff space and T is a topological group which leaves an end point e of X fixed. In [3], Wallace proved that if T is cyclic and X is a locally connected continuum, T has a fixed point other than e. Then in [4], Wallace asked the following question: If X is a peano continuum and T is compact or abelian, then does T have a fixed point other than e?

In [5] Wang showed that if T is compact, and X is arcwise connected, then T has a fixed point other than e. Then Chu [1] showed that T has infinitely many fixed points. Chu began the study of the abelian case in [2].

In this paper we show by example that X may be a peano continuum and T may be a countably generated abelian group which has e for its only fixed point, §2. Thus in general the answer to Wallace's question in the abelian case is no. However, if T is a generative group, and X is compact and arcwise connected, then T has a fixed point other than e, §3. We also show by example that T may be a finitely generated nonabelian group which has e for its only fixed point, §4.

2. Let $\{P_n; n \geqq 1\}$ be a sequence of points which lie on a line; P_{n+1} lies to the right of P_n and the distance from P_{n+1} to P_n is $1/2^n$. The limit of the $\{P_n\}$ is denoted by e. Construct a sequence of sets $\{X_n; n \geqq 1\}$ as follows: X_1, X_2, and X_3 are shown in Figure 1. In general, if $n > 1$, $X_n = A_n \cup B_n$, where A_n is the union of X_{n-1} and the line segment from P_{n-1} to P_n, and B_n is a simplicial replica of A_n; we require that B_n lies in a square, one of whose sides is the line segment from P_{n-1} to P_n, and the intersection of B_n and A_n is exactly P_n. The peano continuum X is the union of e and all the X_n, where X has the usual topology of the plane. A vertex of X is a point other than e where the space "branches." X contains countably many simplicial replicas $\{X_n^{(k)}\}$ of X_n for each n. These complexes contain $2^{n+1}-1$ vertices and are of the form $X_n^{(k)} = A_n^{(k)} \cup B_n^{(k)}$ where $A_n^{(k)}$ and $B_n^{(k)}$ are simplicial replicas of A_n and B_n, respectively. f_n is the homeomorphism: $X \rightarrow X$ which permutes $A_n^{(k)}$ and $B_n^{(k)}$, $k \geqq 1$, simplicially in a

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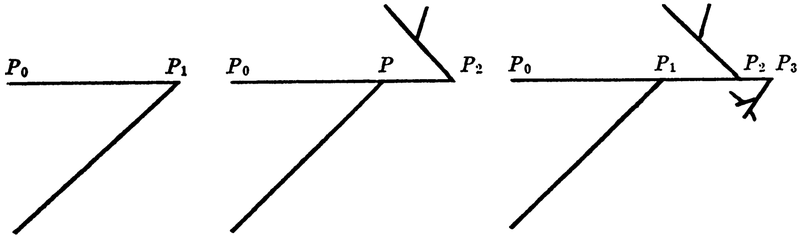

(a) X_1 (b) X_2 (c) X_3

FIGURE 1

natural fashion, and which leaves all other points of X, including e fixed.

The fact that the elements of the sequence $\{f_n\}$ commute with each other is a consequence of the following facts:

(1) If $i<j$, $f_i(X_j)\subset X_j$.

(2) For every $i\geqq 1$, $f_i^2=\text{identity}$.

(3) If $i<j$, and if h_1 and h_2 are two vertices which are permuted by f_i, then the vertices f_jh_1 and f_jh_2 are also permuted by f_i.

All the fixed points of f_n lie to the right of P_n for each n. Hence the f_i can have no fixed point, other than e, in common.

If T denotes the discrete abelian group generated by the $\{f_n\}$, then T has e for its only fixed point.

3. Theorem 1 below is a restatement of a result of Wallace [**3**] and Theorem 2 is equivalent to a result of Wang [**5**].

THEOREM 1. *Let X be a nontrivial Hausdorff continuum and $t\colon X\to X$ be a homeomorphism which leaves an end point e of X fixed. Let A and B be subcontinua of X such that $A\cap B=\{z\}$, where $z\neq e$, $e\in A$, and $X=A\cup B$. If there exists $y\in B$ such that $\{y, t^{-1}y\}\subset B$, either $tz=z$, or else one of the following must hold:*

(a) *For each integer $n\geqq 0$, $t^{n-1}A\subset t^nA$ and $t^nB\subset t^{n-1}B$. Set*

$$K = \mathrm{Cl}(\bigcup\{t^nA;\, n \geqq 0\}), \qquad L = \bigcap\{t^nB;\, n \geqq 0\}.$$

Then K and L are t-invariant continua and $X=K\cup L$.

(b) *For each integer $n\geqq 0$, $t^{-(n-1)}A\subset t^{-n}A$ and $t^{-n}B\subset t^{-(n-1)}B$. Set*

$$K = \mathrm{Cl}(\bigcup\{t^{-n}A;\, n \geqq 0\}), \qquad L = \bigcap\{t^{-n}B;\, n \geqq 0\}.$$

Then K and L are t-invariant continua and $X=K\cup L$.

THEOREM 2. *Let (X, T, π) be a topological transformation group, where X is an arcwise connected Hausdorff space and T leaves an end* *point e of X fixed. Then if there is a nonempty T-invariant closed subset of X which does not contain e, T has a fixed point other than e.*

LEMMA 1. *Let X be a nontrivial arcwise connected compact Hausdorff space and* $t_1, \cdots, t_n$ *be commuting homeomorphisms*: $X \to X$. *Then if each of the* t_i *leaves an end point e of X fixed, then the* t_i *have another fixed point in common.*

PROOF. The proof for the case $n=1$ is obtained by combining Theorems 1 and 2, see Chu [2]. We proceed by induction. Assume the theorem is true for $n=k$, with $k \geqq 1$. Let z_j be a fixed point common to $t_1, \cdots, t_{j-1}, t_{j+1}, \cdots, t_{k+1}$, where $z_j \neq e$ and $1 \leqq j \leqq k+1$. Set $A = \{z_1, \cdots, z_{k+1}\}$. Since e is an end point and $e \notin A$, we may find subcontinua A_1 and B_1 of X for which $X = A_1 \cup B_1$, $e \in A_1$, $A \subset B_1$, $A_1 \cap B_1 = \{x\}$, where $x \neq e$. If $t_i x = x$ for $i = 1, \cdots, k+1$, we are through. Otherwise we may assume without loss of generality that $t_1 x \neq x$. Thus t_1, A_1, and B_1 satisfy the hypothesis of Theorem 1, and we may assume that part (a) of Theorem 1 is applicable. If

$$B_2 = \bigcap \{ t_1^r B_1 ; r \geqq 0 \},$$

then B_2 is t_1-invariant and

$$t_1^r B_1 \subset t_1^{r-1} B_1, \qquad r \geqq 1. \tag{1}$$

Now $z_1 \in B_1$, and therefore $t_1^r z_1 \in t_1^r B_1$; the sequence $\{t_1^r z_1; r \geqq 0\}$ has a cluster point $w \in X$. Because of (1), it is clear that we may take $w \in B_2$. For each $i > 1$, we have $t_i t_1^r z_1 = t_1^r t_i z_1 = t_1^r z_1$ so that $t_i w = w$, $i > 1$. By recursion, define

$$B_j = \bigcap \{ t_{j-1}^r B_{j-1} ; r = 0, \pm 1, \cdots \},$$

for $j = 1, \cdots, k+2$. Then B_j is invariant under $t_1, \cdots, t_{j-1}$, and $w \in B_j$. Thus B_{k+2} is a closed, nonempty subset of X which is invariant under $t_1, \cdots, t_{k+1}$, and $e \notin B_{k+2}$ since

$$B_1 \supset B_2 \supset \cdots \supset B_{k+2}.$$

By Theorem 2, the proof is complete.

The example of §4 shows that Lemma **1** is not true if the t_i do not commute.

LEMMA 2. *Let* (X, T, π) *be a topological transformation group, where X satisfies the hypothesis of Lemma* 1 *and* $T \simeq Z^n R^m$, *where Z is the additive group of all integers with the discrete topology and R is the additive*

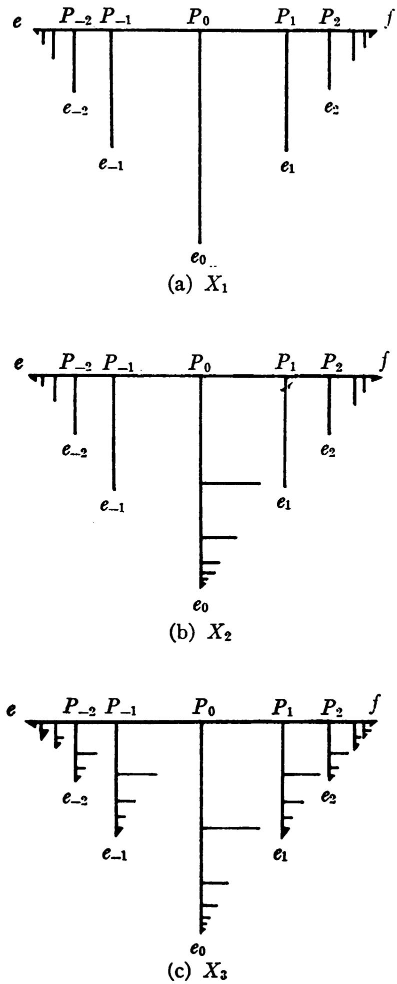

FIGURE 2

group of all real numbers with the usual topology, and m and n are nonnegative integers. Then if T leaves an end point e of X fixed, T has a fixed point other than e.

PROOF. There is a subgroup A of T such that $A \simeq Z^n \times Z^m$ and a compact subset $K \subset T$ for which $T = KA$. By Lemma 1, there is a point $x \in X$, with $x \neq e$, which is fixed under A. Then $Tx = KAx = Kx$ is closed since K is compact. Furthermore $e \notin Tx$. By Theorem 2, the proof is complete.

THEOREM 3. *Let (X, T, π) be a transformation group, where X satisfies the hypothesis of Lemma 1, and T is a generative group (i.e., T is generated by a compact neighborhood of the identity). If T leaves an end point e of X fixed, T has another fixed point.*

The proof follows from Lemma 2 and Theorem 2 since we may write $T \simeq KZ^n R^m$, with K compact, Z and R as in Lemma 2, see Chu [2].

4. Let $\{P_k;\ k \text{ an integer}\}$ be a sequence of points which lie on a line. P_k lies to the left of P_{k+1} and the distance from P_{k+1} to P_k is $1/2^{|k|-1}$ for $k \neq 0$. Let e and f be the limits of the sequences $\{P_k;\ k<0\}$ and $\{P_k;\ k>0\}$ respectively. Define a sequence $\{X_n;\ n \geqq 1\}$ of spaces as follows: X_1, X_2, and X_3 are as in Figure 2. In general, in X_{2n}, that portion of X_{2n} which lies on the line segment from P_0 to e_0 is a simplicial replica of that part of X which lies on the segment from P_0 to f. In X_{2n+1} the part of the space on the segment from P_k to e_k is a replica of the portion of the space which lies on the segment from P_{k+1} to e_{k+1}. The peano continuum X is the union of all the X_n, where X has the topology induced by the plane.

Define homeomorphisms t, s: $X \rightarrow X$ as follows: $t(e) = e$, $t(f) = f$. Let Y_k be the portion of X which is the union of the segment from P_k to P_{k+1} and the part of X which lies on the segment from P_k to e_k. Then Y_k and Y_{k+1} are homeomorphic. Let t "slide" Y_k into Y_{k+1} for each integer k. This defines t. Now the part of X which lies on the segment from P_0 to f is homeomorphic to the part lying on the segment from P_0 to e_0. s permutes these two subspaces and leaves all other points of X fixed. The only fixed points of t are e and f, and $s(f) = e_0$. Hence the discrete group T generated by s and t has e for its only fixed point. The fact that T is not abelian follows from §3.

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