1. Let \((X, T, \pi)\) be a topological transformation group, where \(X\) is a nontrivial Hausdorff space and \(T\) is a topological group which leaves an end point \(e\) of \(X\) fixed. In [3], Wallace proved that if \(T\) is cyclic and \(X\) is a locally connected continuum, \(T\) has a fixed point other than \(e\). Then in [4], Wallace asked the following question: If \(X\) is a peano continuum and \(T\) is compact or abelian, then does \(T\) have a fixed point other than \(e\)?

In [5] Wang showed that if \(T\) is compact, and \(X\) is arcwise connected, then \(T\) has a fixed point other than \(e\). Then Chu [1] showed that \(T\) has infinitely many fixed points. Chu began the study of the abelian case in [2].

In this paper we show by example that \(X\) may be a peano continuum and \(T\) may be a countably generated abelian group which has \(e\) for its only fixed point, §2. Thus in general the answer to Wallace's question in the abelian case is no. However, if \(T\) is a generative group, and \(X\) is compact and arcwise connected, then \(T\) has a fixed point other than \(e\), §3. We also show by example that \(T\) may be a finitely generated nonabelian group which has \(e\) for its only fixed point, §4.

2. Let \(\{P_n; n \geq 1\}\) be a sequence of points which lie on a line; \(P_{n+1}\) lies to the right of \(P_n\) and the distance from \(P_{n+1}\) to \(P_n\) is \(1/2^n\). The limit of the \(\{P_n\}\) is denoted by \(e\). Construct a sequence of sets \(\{X_n; n \geq 1\}\) as follows: \(X_1, X_2, \text{ and } X_3\) are shown in Figure 1. In general, if \(n > 1\), \(X_n = A_n \cup B_n\), where \(A_n\) is the union of \(X_{n-1}\) and the line segment from \(P_{n-1}\) to \(P_n\), and \(B_n\) is a simplicial replica of \(A_n\); we require that \(B_n\) lies in a square, one of whose sides is the line segment from \(P_{n-1}\) to \(P_n\), and the intersection of \(B_n\) and \(A_n\) is exactly \(P_n\). The peano continuum \(X\) is the union of \(e\) and all the \(X_n\), where \(X\) has the usual topology of the plane. A vertex of \(X\) is a point other than \(e\) where the space "branches." \(X\) contains countably many simplicial replicas \(\{X_n^{(k)}\}\) of \(X_n\) for each \(n\). These complexes contain \(2^{n+1} - 1\) vertices and are of the form \(X_n^{(k)} = A_n^{(k)} \cup B_n^{(k)}\) where \(A_n^{(k)}\) and \(B_n^{(k)}\) are simplicial replicas of \(A_n\) and \(B_n\), respectively. \(f_n\) is the homeomorphism: \(X \rightarrow X\) which permutes \(A_n^{(k)}\) and \(B_n^{(k)}\), \(k \geq 1\), simplicially in a

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natural fashion, and which leaves all other points of $X$, including $e$ fixed.

The fact that the elements of the sequence $\{f_n\}$ commute with each other is a consequence of the following facts:

1. If $i < j$, $f_i(X_j) \subset X_j$.
2. For every $i \geq 1$, $f_i^2 =$ identity.
3. If $i < j$, and if $h_1$ and $h_2$ are two vertices which are permuted by $f_i$, then the vertices $f_jh_1$ and $f_jh_2$ are also permuted by $f_i$.

All the fixed points of $f_n$ lie to the right of $P_n$ for each $n$. Hence the $f_i$ can have no fixed point, other than $e$, in common.

If $T$ denotes the discrete abelian group generated by the $\{f_n\}$, then $T$ has $e$ for its only fixed point.

3. Theorem 1 below is a restatement of a result of Wallace [3] and Theorem 2 is equivalent to a result of Wang [5].

**Theorem 1.** Let $X$ be a nontrivial Hausdorff continuum and $t : X \to X$ be a homeomorphism which leaves an end point $e$ of $X$ fixed. Let $A$ and $B$ be subcontinua of $X$ such that $A \cap B = \{z\}$, where $z \neq e$, $e \in A$, and $X = A \cup B$. If there exists $y \in B$ such that $\{y, t^{-1}y\} \subset B$, either $tz = z$, or else one of the following must hold:

(a) For each integer $n \geq 0$, $t^{-n}A \subset t^nA$ and $t^{-n}B \subset t^nB$. Set

$$K = \operatorname{Cl}(\bigcup \{t^nA; n \geq 0\}) \quad \text{and} \quad L = \bigcap \{t^nB; n \geq 0\}.$$ 

Then $K$ and $L$ are $t$-invariant continua and $X = K \cup L$.

(b) For each integer $n \geq 0$, $t^{-(n-1)}A \subset t^{-n}A$ and $t^{-(n-1)}B \subset t^{-n}B$. Set

$$K = \operatorname{Cl}(\bigcup \{t^{-n}A; n \geq 0\}) \quad \text{and} \quad L = \bigcap \{t^{-n}B; n \geq 0\}.$$ 

Then $K$ and $L$ are $t$-invariant continua and $X = K \cup L$.

**Theorem 2.** Let $(X, T, \pi)$ be a topological transformation group, where $X$ is an arcwise connected Hausdorff space and $T$ leaves an end
point e of X fixed. Then if there is a nonempty T-invariant closed subset of X which does not contain e, T has a fixed point other than e.

**Lemma 1.** Let X be a nontrivial arcwise connected compact Hausdorff space and $t_1, \ldots, t_n$ be commuting homeomorphisms: $X \to X$. Then if each of the $t_i$ leaves an end point e of X fixed, then the $t_i$ have another fixed point in common.

**Proof.** The proof for the case $n=1$ is obtained by combining Theorems 1 and 2, see Chu [2]. We proceed by induction. Assume the theorem is true for $n=k$, with $k \geq 1$. Let $z_j$ be a fixed point common to $t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_k$, where $z_j \neq e$ and $1 \leq j \leq k+1$. Set $A = \{z_1, \ldots, z_{k+1}\}$. Since e is an end point and $e \notin A$, we may find subcontinua $A_1$ and $B_1$ of X for which $X = A_1 \cup B_1$, $e \notin A_1$, $A \subset B_1$, $A_1 \cap B_1 = \{x\}$, where $x \neq e$. If $t_i x = x$ for $i = 1, \ldots, k+1$, we are through. Otherwise we may assume without loss of generality that $t_1 x \neq x$. Thus $t_1, A_1$, and $B_1$ satisfy the hypothesis of Theorem 1, and we may assume that part (a) of Theorem 1 is applicable. If

$$B_2 = \bigcap \{t_i^r B_1; r \geq 0\},$$

then $B_2$ is $t_1$-invariant and

$$t_1 B_1 \subset t_1^{-1} B_1, \quad r \geq 1. \tag{1}$$

Now $z_1 \in B_1$, and therefore $t_i^r z_1 \in t_i^r B_1$; the sequence $\{t_i^r z_1; r \geq 0\}$ has a cluster point $w \in X$. Because of (1), it is clear that we may take $w \in B_2$. For each $i > 1$, we have $t_i t_i z_1 = t_i t_i z_1 = t_i z_1$ so that $t_i w = w$, $i > 1$. By recursion, define

$$B_j = \bigcap \{t_{j-1}^r B_{j-1}; r = 0, \pm 1, \ldots\},$$

for $j = 1, \ldots, k+2$. Then $B_j$ is invariant under $t_1, \ldots, t_{j-1}$, and $w \in B_j$. Thus $B_{k+2}$ is a closed, nonempty subset of X which is invariant under $t_1, \ldots, t_{k+1}$, and $e \notin B_{k+2}$ since

$$B_1 \supset B_2 \supset \cdots \supset B_{k+2}.$$ 

By Theorem 2, the proof is complete.

The example of §4 shows that Lemma 1 is not true if the $t_i$ do not commute.

**Lemma 2.** Let $(X, T, \pi)$ be a topological transformation group, where $X$ satisfies the hypothesis of Lemma 1 and $T \cong \mathbb{Z}^n R^m$, where $\mathbb{Z}$ is the additive group of all integers with the discrete topology and $R$ is the additive
group of all real numbers with the usual topology, and $m$ and $n$ are nonnegative integers. Then if $T$ leaves an end point $e$ of $X$ fixed, $T$ has a fixed point other than $e$.

Proof. There is a subgroup $A$ of $T$ such that $A \cong \mathbb{Z}^n \times \mathbb{Z}^m$ and a compact subset $K \subset T$ for which $T = KA$. By Lemma 1, there is a point $x \in X$, with $x \neq e$, which is fixed under $A$. Then $Tx = KAx = Kx$ is closed since $K$ is compact. Furthermore $e \notin Tx$. By Theorem 2, the proof is complete.
Theorem 3. Let \((X, T, \pi)\) be a transformation group, where \(X\) satisfies the hypothesis of Lemma 1, and \(T\) is a generative group (i.e., \(T\) is generated by a compact neighborhood of the identity). If \(T\) leaves an end point \(e\) of \(X\) fixed, \(T\) has another fixed point.

The proof follows from Lemma 2 and Theorem 2 since we may write \(T = KZ^nR^m\), with \(K\) compact, \(Z\) and \(R\) as in Lemma 2, see Chu [2].

4. Let \(\{P_k; k\ \text{an integer}\}\) be a sequence of points which lie on a line. \(P_k\) lies to the left of \(P_{k+1}\) and the distance from \(P_{k+1}\) to \(P_k\) is \(1/2^{|k|-1}\) for \(k \neq 0\). Let \(e\) and \(f\) be the limits of the sequences \(\{P_k; k < 0\}\) and \(\{P_k; k > 0\}\) respectively. Define a sequence \(\{X_n; n \geq 1\}\) of spaces as follows: \(X_1, X_2,\) and \(X_3\) are as in Figure 2. In general, in \(X_{2n}\), that portion of \(X_{2n}\) which lies on the line segment from \(P_0\) to \(e_0\) is a simplicial replica of that part of \(X\) which lies on the segment from \(P_0\) to \(f\). In \(X_{2n+1}\) the part of the space on the segment from \(P_k\) to \(e_k\) is a replica of the portion of the space which lies on the segment from \(P_{k+1}\) to \(e_{k+1}\). The peano continuum \(X\) is the union of all the \(X_n\), where \(X\) has the topology induced by the plane.

Define homeomorphisms \(t, s: X \to X\) as follows: \(t(e) = e, t(f) = f\). Let \(Y_k\) be the portion of \(X\) which is the union of the segment from \(P_k\) to \(P_{k+1}\) and the part of \(X\) which lies on the segment from \(P_k\) to \(e_k\). Then \(Y_k\) and \(Y_{k+1}\) are homeomorphic. Let \(t\) "slide" \(Y_k\) into \(Y_{k+1}\) for each integer \(k\). This defines \(t\). Now the part of \(X\) which lies on the segment from \(P_0\) to \(f\) is homeomorphic to the part lying on the segment from \(P_0\) to \(e_0\). \(s\) permutes these two subspaces and leaves all other points of \(X\) fixed. The only fixed points of \(t\) are \(e\) and \(f\), and \(s(f) = e_0\). Hence the discrete group \(T\) generated by \(s\) and \(t\) has \(e\) for its only fixed point. The fact that \(T\) is not abelian follows from §3.

References