REMARKS ON A QUADRATIC FORM OF DUREN AND SCHIFFER[1]

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Consider an analytic function $f(z)$, schlicht in the unit circle, and normalized so that it has a Taylor series of the form

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots .$$

Bieberbach [1] proved that for any such function $|a_2| \leqq 2$ and he further conjectured that $|a_n| \leqq n$. Obviously, this estimate is the best possible, since the Koebe function, $f(z) = z/(1-z)^2$, is schlicht and has the Taylor series

$$z/(1-z)^2 = z + 2z^2 + \cdots + nz^n + \cdots .$$

So far, Bieberbach's conjecture has been verified only for a_3 (Loewner [5]) and a_4 (Garabedian and Schiffer [4]). Recent numerical experiments by George Ross [6] on a_6 indicate that the conjecture is also true in that case, but so far a full proof has not been obtained. That result and this report are both part of a program at the Courant Institute of Mathematical Sciences for studying the Bieberbach conjecture with the aid of a large computer.

A general approach to this problem has been taken by Duren and Schiffer. In a recent paper [2] they use the calculus of variations to develop a formula for the second variation of a_n for a restricted class of functions $f(z)$ near the Koebe function. This second variation is written as an infinite-dimensional quadratic form. It is necessary, but not sufficient, that the quadratic form be positive-definite for the Bieberbach conjecture to be true. Duren and Schiffer [2] give a direct proof of the positive-definiteness of the quadratic form for a_5 through a_9. However, they found no obvious way for continuing their result to higher values of n.

Our aim is to examine the matrix associated with the Duren and Schiffer quadratic form and to check numerically that it is positive-definite in the range $2 \leqq n \leqq 300$. For odd $n = 2m+1$ the quadratic form is

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$$
\begin{aligned}
S_{2m+1} = \sum_{k=1}^{m} \sum_{\nu=-\infty}^{\infty} \nu(x_{\nu+k} - x_{\nu-k})^2 + \sum_{k=1}^{m} (2m+1-k) \sum_{\alpha+\beta=2k} x_\alpha x_\beta \\
+ 2\sum_{k=2}^{m-1} (m-k) \sum_{\alpha+\beta=2k-1} x_\alpha x_\beta
\end{aligned}
\tag{1}
$$

while for even $n=2m$ it is

$$
\begin{aligned}
S_{2m} = \sum_{k=0}^{m-1} \sum_{\nu=-\infty}^{\infty} (2\nu+1)(x_{\nu+k+1} - x_{\nu-k})^2 + \sum_{k=1}^{m-1} 4(m-k) \sum_{\alpha+\beta=2k} x_\alpha x_\beta \\
+ \sum_{k=1}^{m-1} (4m-2-2k) \sum_{\alpha+\beta=2k+1} x_\alpha x_\beta
\end{aligned}
\tag{2}
$$

where $x_\nu = 0$ for $\nu \leqq 0$ in both cases. Equation (1) may be rewritten as

$$
S_{2m+1} = S'_{2m+1} + \sum_{k=1}^{m} \sum_{\nu=2m-k}^{\infty} \nu(x_{\nu+k} - x_{\nu-k})^2 \tag{3}
$$

where

$$
\begin{aligned}
S'_{2m+1} = \sum_{k=1}^{m} \sum_{\nu=1-k}^{2m-k-1} \nu(x_{\nu+k} - x_{\nu-k})^2 + \sum_{k=1}^{m} (2m+1-k) \sum_{\alpha+\beta=2k} x_\alpha x_\beta \\
+ 2\sum_{k=2}^{m-1} (m-k) \sum_{\alpha+\beta=2k+1} x_\alpha x_\beta .
\end{aligned}
\tag{4}
$$

Similarly, equation (2) may be rewritten as

$$
S_{2m} = S'_{2m} + \sum_{k=0}^{m-1} \sum_{\nu=2m-k-2}^{\infty} (2\nu+1)(x_{\nu+k+1} - x_{\nu-k})^2 \tag{5}
$$

where

$$
\begin{aligned}
S'_{2m} = \sum_{k=0}^{m-1} \sum_{\nu=-k}^{2m-k-3} (2\nu+1)(x_{\nu+k+1} - x_{\nu-k})^2 + 4\sum_{k=1}^{m-1} (m-k) \sum_{\alpha+\beta=2k} x_\alpha x_\beta \\
+ \sum_{k=1}^{m-1} (4m-1-2k) \sum_{\alpha+\beta=2k+1} x_\alpha x_\beta .
\end{aligned}
\tag{6}
$$

Positive-definiteness of S'_{2m+1} and S'_{2m} clearly implies that of S_{2m+1} and S_{2m}. Since the order of the matrices associated with the primed quadratic forms is now finite, they can be diagonalized numerically and the sign of their diagonal elements can be ascertained. To achieve this, two different FORTRAN programs were written for the IBM7094 and the CDC6600 at the AEC Computing and Applied

Mathematics Center of New York University. One program diagonalized the matrix associated with S_n' by means of a nonorthogonal transformation. This program was run for $n=3, \cdots, 300$ and verified that these matrices S_n' are indeed positive-definite.

The second program found the minimum eigenvalue of the matrix associated with S_n'. The minimum eigenvalue of S_n' was computed for $n=3, \cdots, 150$. For S'_{2m+1} it was found experimentally that the minimum eigenvalue, $\lambda_{\min}$, lies in the range

$$m < \lambda_{\min} < m + 2.$$

However, for S'_{2m} the more striking numerical result was discovered that

$$\lambda_{\min} = 2m - 1.$$

Therefore, we conjecture that this result is exact for all S'_{2m}. We have been able to prove that $2m-1$ is an exact eigenvalue corresponding to the eigenvector $(-2m+3, 2, \cdots, 2)$, but we have not yet been able to show that it is actually the minimum eigenvalue.

References

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