LINEAR DIFFERENCE OPERATORS ON
PERIODIC FUNCTIONS

OTTO PLAAT

Let \( p > 0 \) and \( B \) the Banach space of continuous functions \( f: \mathbb{R}^1 \rightarrow \mathbb{R}^1 \) of period \( p \), with \( \|f\| = \max \{ |f(x)| ; 0 \leq x \leq p \} \). Let \( a \in B \), \( a(x) > 0 \) for all \( x \), and let \( t \) be a real number. Define the bounded linear operator \( L: B \rightarrow B \) by \( Lf(x) = f(x+t) - a(x)f(x) \). We shall obtain results concerning the solutions in \( B \) of the equation \( Lf = g \). We say that \( L \) is regular if it is one-to-one and onto \( B \), and that it is singular otherwise. The limit \( \lim_{n \to \infty} \left( \frac{1}{n} \sum_{k=0}^{n-1} f(x+kt) \right) \), where \( f \in B \), will be denoted by \( \sigma_f(x) \). If \( t/p \) is irrational, the existence of the limit follows from the fact that for every \( x \) the sequence \( x_k = x + kt \) (mod \( p \)) is uniformly distributed in \([0, p]\), so that \( \sigma_f(x) = (1/p) \int_0^p f(x) dx \). If \( t/p \) is rational, say \( qt = rp \), with \( q \) and \( r \) integers and \( q > 0 \), then \( \sigma_f(x) = (1/q) \sum_{k=0}^{q-1} f(x+kt) \). Note that \( \sigma_{af}(x) = \sigma_{af}(x) = \sigma_{af}(x+t) \).

The case \( a(x) = 1 \), i.e., the equation

\[
(1) \quad f(x+t) - f(x) = g(x),
\]

was first studied by Euler, whose method was essentially a formal use of Fourier series. The number-theoretic problems which arise via Fourier analysis have been discussed by Wintner [1], who also furnishes an exposition of Euler's work.

The basic result concerning (1) is the following ([2], [3]):

**Theorem 1.** Let \( g \in B \). There exists \( f \in B \) satisfying (1) if and only if \( \sum_{k=0}^{n-1} g(x+kt) \) is bounded (in \( x \) and \( n \)).

The application of this result to the general case rests on a trivial reformulation:

**Theorem 2.** There exists a positive \( b \) in \( B \) satisfying \( a(x) = b(x+t)/b(x) \) if and only if \( \sum_{k=0}^{n-1} \log a(x+kt) \) is bounded.

An explicit formula for the solutions of (1) will also find an application.

**Theorem 3.** The solutions of (1) are given by

\[
f(x) = u(x) - \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} (n-k)g(x+kt),
\]

Received by the editors December 20, 1965.
where \( u \) is an arbitrary function (in \( B \)) of period \( t \).

**Proof.** Let \( f \) be a solution, so that
\[
f(x + (j + 1)t) - f(x + jt) = g(x + jt)
\]
for all integers \( j \). Sum on \( j \):
\[
f(x + kt) - f(x) = \sum_{j=0}^{k-1} g(x + jt).
\]
Now sum on \( k \) and average:
\[
\frac{1}{n} \sum_{k=1}^{n} f(x + kt) - f(x) = \frac{1}{n} \sum_{k=1}^{n} \sum_{j=0}^{k-1} g(x + jt)
\]
\[
= \frac{1}{n} \sum_{k=0}^{n-1} (n - k)g(x + kt).
\]
Since the left side converges as \( n \to \infty \), so does the right side, and we have
\[
\sigma_{tf}(x) - f(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} (n - k)g(x + kt).
\]
We shall denote the limit on the right by \( \alpha_{tg}(x) \). Noting that \( \sigma_{tf} \) has period \( t \), and that the solutions of (1) are determined only to within the addition of arbitrary functions (in \( B \)) of period \( t \), we may suppose without loss of generality that \( \sigma_{tf} = 0 \). Thus \( f = -\alpha_{tg} \), and the addition of an arbitrary \( u \) completes the proof. (If \( t/p \) is irrational, \( u \) is necessarily constant.)

**Remark.** Diliberto [3] has shown via a different argument that for any \( b \in B \) for which \( \alpha_{tb}(x) \) does not vanish, the function
\[
f(x) = -\frac{1}{\sigma_{tb}(x)} \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} b(x + kt) \sum_{j=0}^{k-1} g(x + jt)
\]
is a solution of (1). Setting \( b(x) \equiv 1 \) yields the formula \( f = -\alpha_{tg} \) obtained above. It is clear that (2) does not, in general, produce all solutions of (1), and that different \( b \)'s may produce identical \( f \)'s.

**Theorem 4.** A necessary condition that \( L \) is one-to-one is that
\[
\sum_{k=0}^{n} \log a(x + kt)
\]
is unbounded. If \( t/p \) is irrational, this condition is also sufficient.

**Proof.** Suppose that the sum in question is bounded, so that by Theorem 2 we may write \( a(x) = b(x+t)/b(x) \).
The equation \( Lf = 0 \) thus becomes \( b(x)f(x+t) - b(x+t)f(x) = 0 \).
Since $f = b$ is a nontrivial solution, $L$ is not one-to-one.

Now let $t/p$ be irrational. If $L$ is not one-to-one, there exists $f \neq 0$ such that $f(x+t) = a(x)f(x)$. We show that $f(x)$ cannot vanish. For any $x_0, k$,

$$f(x_0 + (k + 1)t) = a(x_0 + kt)f(x_0 + kt),$$

so that, by elimination,

$$f(x_0 + nt) = f(x_0) \prod_{k=0}^{n-1} a(x_0 + kt)$$

for all $n$. If $f(x_0) = 0$, it follows that $f(x_0 + nt) = 0$ for all $n$. Since the sequence $x_0 + nt \pmod{p}$ is dense in $[0, p]$ and $f$ is continuous, it follows that $f = 0$, a contradiction. Hence there exists a positive $f$ such that $a(x) = f(x+t)/f(x)$, and the conclusion now follows from Theorem 2.

**Remark.** A sufficient condition in the rational case $qt = rp$ is easily seen to be that the sum is unbounded on every interval—in other words, that $\{x; \sum_{k=0}^{n-1} \log a(x + kt) \neq 0\}$ is dense.

The following is a generalization of Theorems 1 and 3.

**Theorem 5.** Let $\sum_{k=0}^{n-1} \log a(x + kt)$ be bounded. Then the equation $Lf = g$ has a solution if and only if $\sum_{k=0}^{n-1} \exp [\alpha_t \log a(x + (k + 1)t)] g(x + kt)$ is bounded. The solutions are given by

$$f(x) = \exp[-\alpha_t \log a(x)][u(x) - \alpha_t [\exp[\alpha_t \log a(x + t)]g(x)]],$$

where $u$ is an arbitrary function (in $B$) of period $t$.

**Proof.** By Theorems 1 and 3, we may set

$$c(x) = \exp[\alpha_t \log a(x)]$$

so that $a(x) = c(x)/c(x + t)$. The equation $Lf = g$ thus becomes

$$c(x + t)f(x + t) - c(x)f(x) = c(x + t)g(x).$$

It is therefore solvable if and only if $\sum_{k=0}^{n-1} c(x + (k + 1)t)g(x + kt)$ is bounded, which is the stated condition. Referring again to Theorem 3, we find that $- (1/c(x)) \alpha_t [c(x + t)g(x)]$ is a solution, to which may be added an arbitrary solution of $Lf = 0$. The solutions of the latter are of the form $u(x)/c(x)$. This concludes the proof.

**Remark.** The content of Theorem 5 is clarified by noting that the hypothesis implies the factorization $L = QRS$, where

$$Qf(x) = (1/c(x + t))f(x), \quad Rf(x) = f(x + t) - f(x), \quad Sf(x) = c(x)f(x).$$
Theorem 6. If $L$ is onto $B$, it is one-to-one.

Proof. Consider first the rational case $qt=rp$. If $L$ is onto, there is an $f$ satisfying $f(x+t)-a(x)f(x)=1$. Translating by $kt$, $k=1, 2, \ldots, q-1$, we find by elimination that

$$\left[1 - \prod_{k=0}^{q-1} a(x + kt)\right] f(x) = \sum_{n=0}^{q-1} \prod_{k=n+1}^{q-1} a(x + kt),$$

so that $\prod_{k=0}^{q-1} a(x + qt) \neq 1$, all $x$. Hence, $\sum_{k=0}^{q-1} \log a(x + qt)$ does not vanish. The remark following Theorem 4 completes the proof in the rational case.

Now suppose $t/p$ is irrational. If $L$ is not one-to-one, it follows from Theorem 4 that the hypothesis of Theorem 5 is satisfied, and obviously the equation $L_f = 1$ has no solution.

Theorem 7. A necessary and sufficient condition that $L$ is regular is that $\sigma \log a(x)$ does not vanish.

Proof. We dispose first of the rational case $qt=rp$. If $L$ is regular, we see, as in the proof of Theorem 6, that $\sum_{k=0}^{q-1} \log a(x + qt)$ does not vanish. But this is precisely the condition $\sigma \log a(x) \neq 0$. The sufficiency of the condition follows in the same way; we obtain the formula

$$L^{-1}g(x) = \frac{\sum_{n=0}^{q-1} g(x + nt) \prod_{k=n+1}^{q-1} a(x + kt)}{1 - \prod_{k=0}^{q-1} a(x + kt)}.$$

We now establish the sufficiency of the condition in the general case. Define operators $T$ and $M$ by $Tf(x) = f(x+t)$, $Mf(x) = a(x-t)f(x-t)$, so that $L = T(I-M)$. It suffices to show that $I-M$ is regular. We shall, in fact, show that $\|M^n\| < 1$ for some (positive or negative) integer $n$. Henceforth let $n$ be a positive integer, and suppose first that $\sigma \log a(x) < 0$. Then

$$M^n f(x) = f(x - nt) \prod_{k=1}^{n} a(x - kt), \quad \|M^n\| = \max_x \prod_{k=1}^{n} a(x - kt),$$

and

$$\log \|M^n\| = \max_x \sum_{k=1}^{n} \log a(x - kt).$$

The hypothesis implies that if $n$ is sufficiently large, then $\sum_{k=1}^{n} \log a(x - kt) < 0$ for all $x$, and therefore $\|M^n\| < 1$ for such $n$. Thus $I-M$ is regular.
Next, suppose that $\sigma \log a(x) > 0$. Since
\[
M^{-n}f(x) = f(x + nt) \left[ \prod_{k=0}^{n-1} a(x + kt) \right]^{-1},
\]
and
\[
\|M^{-n}\| = \left[ \min_x \prod_{k=0}^{n-1} a(x + kt) \right]^{-1},
\]
we have that
\[
\log\|M^{-n}\| = -\min_x \sum_{k=0}^{n-1} \log a(x + kt).
\]
The hypothesis now implies that $\sum_{k=0}^{n-1} \log a(x+kt) > 0$ for all $x$, and for $n$ sufficiently large. Hence $\|M^{-n}\| < 1$ for such $n$, and it follows that $M^{-1} - I$ is regular. Since $I - M = M(M^{-1} - I)$, the sufficiency is proved.

The necessity of the condition in the irrational case will be established by showing that if $\sigma \log a(x) = 0$, there exists a sequence of singular operators which converges to $I$ in norm. Recall that in this case $\sigma \log a(x) = (1/p)\int_0^p \log a(x)dx$. We shall need the following remark: If $t/p$ is irrational and $g(x)$ is a trigonometric polynomial (t.p.) of period $p$ and mean value 0, then there exists a t.p. $h(x)$ of period $p$ such that $h(x+t) - h(x) = g(x)$. The proof is a simple computation. In fact, if
\[
g(x) = \sum_{0 < |k| \leq N} a_k \exp\left(\frac{2\pi ikx}{p}\right), \quad a_k = a_{-k},
\]
then
\[
h(x) = \sum_{0 < |k| \leq N} \frac{a_k}{\exp(2\pi ik/p) - 1} \exp(2\pi ikx/p)
\]
is the desired t.p.

By the Weierstrass approximation theorem, there exists a sequence of t.p.'s $g_n(x)$ of period $p$ converging uniformly to $\log a(x)$. Since, by hypothesis, $\int_0^p \log a(x)dx = 0$, we may suppose, modifying the $g_n$ if necessary, that $\int_0^p g_n(x)dx = 0$. Let $h_n(x)$ be a sequence of t.p.'s satisfying $h_n(x+t) - h_n(x) = g_n(x)$. Define the operators
\[
L_nf(x) = f(x + t) - \exp[g_n(x)]f(x), \quad Rf(x) = f(x + t) - f(x),
\]
\[
V_nf(x) = \exp[h_n(x + t)]f(x), \quad W_nf(x) = \exp[-h_n(x)]f(x).
\]
It is readily verified that $L_n = V_nRW_n$. $R$ is obviously singular, whence $L_n$ is singular. But $L_n$ converges to $L$ in norm. Hence $L$ is singular.

**Corollary.** If $\sigma_1 \log a(x) < 0$, then

$$L^{-1}g(x) = g(x - l) + \sum_{n=1}^{\infty} g(x - (n + 1)t) \prod_{k=1}^{n} a(x - (k + 1)t),$$

and if $\sigma_1 \log a(x) > 0$, then

$$L^{-1}g(x) = -\sum_{n=0}^{\infty} \left[ g(x + nt) \prod_{k=0}^{n} a(x + kt) \right],$$

the series converging absolutely and uniformly.

**Proof.** We have the expansion $L^{-1} = (I - M)^{-1}T^{-1} = \sum_{n=0}^{\infty} M^nT^{-1}$ in the first case, and

$$L^{-1} = -(I - M^{-1})^{-1}M^{-1}T^{-1} = -\sum_{n=0}^{\infty} M^{-n-1}T^{-1}$$

in the second. Absolute convergence follows from the norm estimate on $M^n(M^{-n})$ given in the proof of the theorem. These yield the above expansions for $L^{-1}g(x)$.

**References**