Let $A$ be a commutative semisimple regular Banach algebra with identity $1$ and maximal ideal space $\mathfrak{M}$. For simplicity, we identify $A$ and $A^\wedge$, the Gelfand representatives of $A$ in $C(\mathfrak{M})$. Thus, $A$ is an algebra of continuous functions containing the constants on the compact Hausdorff space $\mathfrak{M}$. A well known consequence of the regularity of $A$ is the fact that any element of $C(\mathfrak{M})$ which is locally in $A$ is actually in $A$. (A function $f$ on $\mathfrak{M}$ is said to be locally in $A$ if for each $x$ in $\mathfrak{M}$ there exists a neighborhood $U$ of $x$ and element $a$ in $A$ such that $f=a$ on $U$.) The purpose of this note is to prove a related result for certain subalgebras $B$ of $A$. We say that a subalgebra $B$ of $C(\mathfrak{M})$ separates the points of $\mathfrak{M}$ if to each $x, y$ in $\mathfrak{M}$, $x \neq y$, there exists $b$ in $B$ such that $b(x)=0$ and $b(y)=1$. (If $1\in B$, this is the same as the usual definition of “separating”.)

**Theorem.** Let $B$ be a subalgebra (not necessarily closed) of the regular algebra $A$, which separates the points of $\mathfrak{M}$, and suppose that every element of $A$ is locally in $B$. Then $B=A$.

[Applied, in particular, to $A=C(X)$, $X$ compact Hausdorff, this says that any separating subalgebra $B$ of $C(X)$ which yields the same “germs” as $C(X)$ at each $x$ in $X$ is necessarily all of $C(X)$.

Proof. For any $a$ in $A$ and $x$ in $\mathfrak{M}$ there is an element $b$ of $B$ with $b=a$ on some neighborhood $U$ of $x$. Thus, by compactness, there exist elements $b_1, \ldots, b_n$ of $B$ and an open covering $U_1, \ldots, U_n$ of $\mathfrak{M}$ for which $a=b_i$ on $U_i$. It will suffice to prove that subordinate to such a covering there exists a “partition of unity” $\{e_1, \ldots, e_n\}$ in $B$ (i.e., each $e_i$ vanishes off $U_i$, and $\sum e_i=1$). Indeed, we then observe that $a=\sum e_i b_i$ is in $B$.

Suppose that $x$ and $y$ are distinct points of $\mathfrak{M}$. By hypothesis, there exists $b$ in $B$ with $b(x)=0$ and $b(y)=1$, so $0 \in b(W)$ for some compact neighborhood $W$ of $y$. Let $kW$ denote the ideal $\{a: a\in A, a(W)=0\}$. Since $A$ is regular, $W$ is the maximal ideal space of the quotient algebra $A/kW$ and the corresponding Gelfand representation is defined by $a+kW \rightarrow a|_W$. Thus, $b$ gives rise to an invertible element of $A/kW$, so $ab=1$ on $W$ for some $a$ in $A$. But the element $a$ coincides with some $b'$ in $B$ on a neighborhood of $y$, so $bb'=1$ near $y$ while $bb'(x)=0$. Now the element $1-bb'$ is in $A$, vanishes in a neighborhood of $y$ and is 1

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at \( x \), so the same argument, applied to \( 1 - bb' \) and \( x, y \) interchanged, shows that there exists \( b'' \) in \( B \) such that \( b''(1 - bb') = 1 \) in a neighborhood of \( x \). Thus, \( e = b'' - b''bb' \) is in \( B \), vanishes near \( y \) (since \( 1 - bb' \) does) and is 1 near \( x \). By a well-known argument [2] the existence of such elements \( e \) of \( B \) shows that \( B \) is a normal algebra of functions on \( \mathfrak{M} \) (in the obvious sense) so that the desired partitions of unity can be obtained as in [2].

Note that we need to assume that the points of \( \mathfrak{M} \) at which the elements of \( A \) belong locally to \( B \) comprise all of \( \mathfrak{M} \): Consider the sub-algebra \( B \) of \( C([0, 1]) \) consisting of those functions which coincide with a polynomial near 0. Also, \( B \) must be an algebra and not just a subspace, as is shown by the subspace \( B \) of \( C([-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]) \) consisting of all functions of the form \( f + c \) where \( c \) is a constant and \( f \) as an odd function.

As is well known [1], if \( A \) is a sup norm algebra, elements of \( C(\mathfrak{M}) \) belonging locally to \( A \) need not belong to \( A \); whether the analogue of our result is valid for sup norm algebras \( A \) remains an open question.

**Bibliography**


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