A REMARK ON REGULAR BANACH ALGEBRAS

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Let A be a commutative semisimple regular Banach algebra with identity 1 and maximal ideal space $\mathfrak{M}$. For simplicity, we identify A and $A^\wedge$, the Gelfand representatives of A in $C(\mathfrak{M})$. Thus, A is an algebra of continuous functions containing the constants on the compact Hausdorff space $\mathfrak{M}$. A well known consequence of the regularity of A is the fact that any element of $C(\mathfrak{M})$ which is locally in A is actually in A. (A function f on $\mathfrak{M}$ is said to be locally in A if for each x in $\mathfrak{M}$ there exists a neighborhood U of x and element a in A such that $f=a$ on U.) The purpose of this note is to prove a related result for certain subalgebras B of A. We say that a subalgebra B of $C(\mathfrak{M})$ separates the points of $\mathfrak{M}$ if to each x, y in $\mathfrak{M}$, $x\neq y$, there exists b in B such that $b(x)=0$ and $b(y)=1$. (If $1\in B$, this is the same as the usual definition of "separating".)

THEOREM. *Let B be a subalgebra (not necessarily closed) of the regular algebra A, which separates the points of $\mathfrak{M}$, and suppose that every element of A is locally in B. Then $B=A$.*

[Applied, in particular, to $A=C(X)$, X compact Hausdorff, this says that any separating subalgebra B of $C(X)$ which yields the same "germs" as $C(X)$ at each x in X is necessarily all of $C(X)$.]

PROOF. For any a in A and x in $\mathfrak{M}$ there is an element b of B with $b=a$ on some neighborhood U of x. Thus, by compactness, there exist elements $b_1, \cdots, b_n$ of B and an open covering $U_1, \cdots, U_n$ of $\mathfrak{M}$ for which $a=b_i$ on U_i. It will suffice to prove that subordinate to such a covering there exists a "partition of unity" $\{e_1, \cdots, e_n\}$ in B (i.e., each e_i vanishes off U_i, and $\sum e_i=1$). Indeed, we then observe that $a=\sum e_i b_i$ is in B.

Suppose that x and y are distinct points of $\mathfrak{M}$. By hypothesis, there exists b in B with $b(x)=0$ and $b(y)=1$, so $0\notin b(W)$ for some compact neighborhood W of y. Let kW denote the ideal $\{a: a\in A,\ a(W)=0\}$. Since A is regular, W is the maximal ideal space of the quotient algebra A/kW and the corresponding Gelfand representation is defined by $a+kW\to a|_W$. Thus, b gives rise to an invertible element of A/kW, so $ab=1$ on W for some a in A. But the element a coincides with some b' in B on a neighborhood of y, so $bb'=1$ near y while $bb'(x)=0$. Now the element $1-bb'$ is in A, vanishes in a neighborhood of y and is 1

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at x, so the same argument, applied to $1-bb'$ and x, y interchanged, shows that there exists b'' in B such that $b''(1-bb')=1$ in a neighborhood of x. Thus, $e=b''-b''bb'$ is in B, vanishes near y (since $1-bb'$ does) and is 1 near x. By a well-known argument [2] the existence of such elements e of B shows that B is a normal algebra of functions on $\mathfrak{M}$ (in the obvious sense) so that the desired partitions of unity can be obtained as in [2].

Note that we need to assume that the points of $\mathfrak{M}$ at which the elements of A belong locally to B comprise all of $\mathfrak{M}$: Consider the subalgebra B of $C([0, 1])$ consisting of those functions which coincide with a polynomial near 0. Also, B must be an algebra and not just a subspace, as is shown by the subspace B of $C([-1, -\frac{1}{2}]\cup[\frac{1}{2}, 1])$ consisting of all functions of the form $f+c$ where c is a constant and f as an odd function.

As is well known [1], if A is a sup norm algebra, elements of $C(\mathfrak{M})$ belonging locally to A need not belong to A; whether the analogue of our result is valid for sup norm algebras A remains an open question.

Bibliography

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