ANOTHER S-ADMISSIBLE CLASS OF SPACES

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For a topological space X, $S(X)$ denotes the semigroup of all continuous functions mapping X into itself where the binary operation is that of composition. A class of topological spaces was defined in [4] to be S-admissible if for any two spaces X and Y of the class, every isomorphism from $S(X)$ onto $S(Y)$ is induced by a homeomorphism. By this, we mean that for each isomorphism ϕ from $S(X)$ onto $S(Y)$, there exists a homeomorphism h from X onto Y such that $\phi(f)=h \circ f \circ h^{\leftarrow}$ for each f in $S(X)$. Hence, the S-spaces defined in [2] form an S-admissible class of spaces. *We refer to a topological space X as an S^*-space if it is T_1 and for each closed subset F of X and each point p in $X-F$ there exists a function f in $S(X)$ and a point y in X such that $f(x)=y$ for each x in F and $f(p)\neq y$.* Our main purpose here is to prove the following

THEOREM 1. *The class of S^*-spaces is an S-admissible class.*

PROOF. Let X and Y be S^*-spaces and let ϕ be an isomorphism from $S(X)$ onto $S(Y)$. It follows from Theorem 2.4 of [3] that there exists a bijection h from X onto Y such that $\phi(f)=h \circ f \circ h^{\leftarrow}$ for each f in $S(X)$. Since the existence of the bijection h is not hard to prove, we do so here for the sake of completeness. For a point x in X, we use the symbol $\mathbf{x}$ to denote the constant function defined by $\mathbf{x}(y)=x$ for each y in X. Since a function g in $S(X)$ has the property $g \circ f=g$ for each f in $S(X)$ if and only if g is a constant function, it follows that the set of all constant functions of $S(X)$ is precisely the ideal of left zeros of $S(X)$. Then, for any point x in X, $\mathbf{x}$ is a left zero of $S(X)$ and thus $\phi(\mathbf{x})$ is a left zero of $S(Y)$, i.e., $\phi(\mathbf{x})=\mathbf{y}$ for some y in Y. Define $h(x)=y$. Since ϕ maps the ideal of left zeros of $S(X)$ bijectively onto the ideal of left zeros of $S(Y)$, h is a bijection from X onto Y. Using the fact that for any x in X, $\phi(\mathbf{x})=\boldsymbol{h}(\mathbf{x})$, we see that for f in $S(X)$ and y in Y,

$$
\begin{aligned}
(h \circ f \circ h^{\leftarrow})(y) &= h(f(h^{\leftarrow}(y))) = \boldsymbol{h(f(h^{\leftarrow}(y)))}(y) \\
&= \phi(\boldsymbol{f(h^{\leftarrow}(y))})(y) = \phi(f \circ \boldsymbol{h^{\leftarrow}(y)})(y) \\
&= (\phi(f) \circ \phi(\boldsymbol{h^{\leftarrow}(y)}))(y) = (\phi(f) \circ \mathbf{y})(y) = \phi(f)(y).
\end{aligned}
$$

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Thus, $\phi(f)=h\circ f\circ h^{*}$.

Now for any point z in X and any function f in $S(X)$, we let

$$H(z,f) = \{x \in X : f(x) = z\},$$

and we show that

$$h[H(z,f)] = H(h(z), \phi(f)).$$

This is a consequence of the fact that the following statements are successively equivalent. $y\in h[H(z,f)]$, $y=h(x)$ and $f(x)=z$, $\phi(x)=y$ and $\phi(f)\circ\phi(x)=\phi(f\circ x)=\phi(f(x))=\phi(z)$, $\phi(f)\circ y=h(z)$, $\phi(f)(y)=h(z)$, $y\in H(h(z),\phi(f))$.

The proof will therefore be complete when we show that for an S^*-space X, the family of sets of the form $H(z,f)$ is a basis for the closed subsets of X. Since X is T_1, any set of the form $H(z,f)$ is closed. Now let F be a proper closed subset of X and let x be a point of $X-F$. Then there exists a function f_x in $S(X)$ and a point y_x in X such that $f_x(z)=y_x$ for each z in F and $f_x(x)\neq y_x$. It follows that

$$F = \bigcap\{H(y_x, f_x) : x \in X - F\}$$

and the theorem is proved.

The following two results indicate that the class of S^*-spaces is reasonably extensive.

THEOREM 2. *Every* 0-*dimensional Hausdorff space is an* S^*-*space.*

PROOF. Suppose F is a closed subset of X and p is a point in $X-F$. Then there exists a set G which is both open and closed such that $p\in G\subset X-F$. Choose any point q other than p and define $f(x)=p$ for x in G and $f(x)=q$ for x in $X-G$. Then f is continuous, which proves X is an S^*-space.

THEOREM 3. *Every completely regular Hausdorff space containing at least two distinct points which are connected by an arc is an* S^*-*space.*

PROOF. Suppose F is a closed subset of X and $p\in X-F$. Since X is completely regular, there exists a continuous function f mapping X into the closed interval I such that $f(x)=0$ for x in F and $f(p)=1$. Let q_1 and q_2 be two distinct points of X which are connected by an arc. Then there exists a continuous function g from I into X such that $g(0)=q_1$ and $g(1)=q_2$. Therefore, $g\circ f$ is a function in $S(X)$ with the properties $g\circ f(x)=q_1$ for x in F and $g\circ f(p)\neq q_1$.

Let X be any completely regular, Hausdorff space and let $Y=X\cup I$. We assume X and I are disjoint (suitable modifications can be made otherwise). Take as a basis for the open sets of Y those sets which are open either in X or in I. Then by the previous theorem, Y is an S^*-space and we have

COROLLARY 4. *Every completely regular Hausdorff space is a subspace of an S^*-space.*

There exist completely regular Hausdorff spaces which are not S^*-spaces. In fact, Cook [1] has given an example of a compact, metric, one-dimensional, indecomposable continuum K such that $S(K)$ consists precisely of the constant functions and the identity function. K is certainly not an S^*-space. Moreover, no S-admissible class can contain K since any bijection from $S(K)$ onto $S(K)$ which maps the identity onto itself is an isomorphism and only one of these is induced by a homeomorphism. By the previous corollary, K is a subspace of an S^*-space. Thus, the property of being an S^*-space is not hereditary.

An open set G containing a point x in X was defined in [2] to be an S-neighborhood if

(i) G consists of X alone or

(ii) there exists a continuous function f from cl G into X such that $f(x) \neq x$ and $f(y) = y$ for each y in cl $G - G$.

A space X was then defined to be an S-space if each point has a basis of S-neighborhoods. It was shown in [2] that the class of S-spaces is S-admissible. Once more, let K be the space described by Cook, let I be the closed unit interval and topologize $Y = K \cup I$ by taking as a basis for open sets those sets which are open either in X or in I. Choose x in K and let G be any open subset of K containing x such that $K - G$ consists of more than one point. Now suppose there exists a continuous function f mapping $\text{cl}_Y G$ into Y such that $f(x) \neq x$ and $f(y) = y$ for each y in $\text{cl}_Y G - G$. Define a function g from Y into Y by $g(y) = y$ if $y \in Y - G$ and $g(y) = f(y)$ for $y \in G$. Then g is continuous and since K is connected, $g[K]$ is connected. This, together with the facts that $g[K] \cap K \neq \emptyset$, and K and I are both open subsets of Y, implies $g[K] \subseteq K$. This, however, is a contradiction since g is neither the identity function nor a constant function. This implies that the point x has no S-basis and therefore that Y is not an S-space. It follows from Theorem 3, however, that Y is an S^*-space. Therefore, there exist S^*-spaces which are not S-spaces. We do not know if every S-space must be an S^*-space.

We conclude by mentioning that the space described in Example (2.7) of [2] is another example of a space which is not an S^*-space.

References

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