A CONVERSE OF BANACH'S CONTRACTION THEOREM

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1. **Introduction.** Let (S, ρ) be a bounded complete metric space and ϕ a contractive mapping of S into itself; i.e.,

$$\rho(\phi(x), \phi(y)) \leqq \alpha\rho(x, y), \quad \text{where} \quad \alpha \in (0, 1), \quad x, y \in S.$$

Then it follows from a theorem due to Banach that the iterated images, $\phi^n(S)$ of S shrink to the fixed point, a of ϕ. This fact can be written in the form $\bigcap_{n=1}^{\infty} \phi^n(S) = \{a\}$.

Since this formula does not involve the metric and has a topological character, it is therefore natural to ask the following question.

Let S be a compact metrizable topological space and ϕ a continuous mapping of S into itself which has the property that $\bigcap \phi^n(S) = \{a\}$. Is it possible to find a metric $\rho(x, y)$ generating the given topology of S such that the mapping ϕ is contractive with respect to ρ? The answer to this question is affirmative and we will give the construction of the desired metric ρ. It should be mentioned that in the paper [1] a similar problem has been solved for a given abstract set S and a mapping ϕ: $S \to S$ satisfying the condition that each iteration ϕ^n of ϕ has a unique fixed point.

2. **Construction of the metric with respect to which ϕ is nonexpansive.** We will assume S to be a compact metrizable topological space, and denote by $\mathfrak{M}$ the set of all metrics on S generating the given topology of S. The mapping ϕ will be assumed to be continuous on S^* and to satisfy $\bigcap \phi^n(S) = \{a\}$.

2.1. THEOREM 1. *Under the assumptions made on ϕ, there exists in $\mathfrak{M}$ a distance function $\bar{\rho}$ such that $\bar{\rho}(\phi(x), \phi(y)) \leqq \bar{\rho}(x, y)$.*

PROOF. Let us take any $\rho \in \mathfrak{M}$ and define $\bar{\rho}(x, y)$ as follows. $\bar{\rho}(x, y) = \sup_n \rho(\phi^n(x), \phi^n(y))$, for $n = 0, 1, 2, \cdots$. From the definition we have $\bar{\rho}(x, y) \geqq \rho(x, y)$. The triangle inequality follows easily. For let $x, y, z \in S$. From the definition of $\bar{\rho}$ there exists a number n such that $\bar{\rho}(x, z) = \rho(\phi^n(x), \phi^n(z))$, and

$$\bar{\rho}(x, z) \leqq \rho(\phi^n(x), \phi^n(y)) + \rho(\phi^n(y), \phi^n(z)) \leqq \bar{\rho}(x, y) + \bar{\rho}(y, z).$$

In order to prove that $\bar{\rho} \in \mathfrak{M}$ we have only to show that $x_n \to \rho x$ implies $x_n \to \bar{\rho} x$. Let us assume that $\bar{\rho}(x_n, x) \not\to 0$. Then there exists a

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subsequence of $\{x_n\}$, which we denote again by $\{x_n\}$, for which $\bar{\rho}(x_n, x)\to\gamma>0$, where γ is some positive number. From the definition of $\bar{\rho}$ follows the existence of integers $k_1, k_2, \cdots$ such that $\bar{\rho}(x_n, x) =\rho(\phi^{k_n}(x_n), \phi^{k_n}(x))$ for $n=1, 2, \cdots$. We have to consider two cases:

(a) The set $\{k_n\}$ is bounded.

Then there exists a number k in the sequence $\{k_n\}$ which is infinitely repeated and therefore for a suitably selected subsequence we have $\rho(\phi^k(x_n), \phi^k(x))\to\gamma>0$ which is a contradiction because $x_n\to x$ and therefore also $\phi^k(x_n)\to\phi^k(x)$.

(b) The set $\{k_n\}$ is not bounded.

Selecting a suitable subsequence we have the result that

$$\rho(\phi^{k_n}(x_n), \phi^{k_n}(x)) \to \gamma > 0$$

where $\{k_n\}$ is now a monotonically increasing sequence of natural numbers. Because $\phi^{k_n}(x)\in\phi^{k_n}(S)$ and $\bigcap\phi^{k_n}(S)=\{a\}$ we arrive again at a contradiction, which proves our assertion.

3. The construction of a metric ρ^* with respect to which the mapping ϕ is contractive.

3.1. Definition. Let $S=A_0, \phi(S)=A_1, \cdots, \phi^n(S)=A_n \cdots$ and introduce the functions $n(x)$ and $n(x, y)$ as follows:

$$n(x) = \max\{n: x \in A_n\}, \qquad n(x, y) = \min\{n(x), n(y)\}.$$

3.2. Theorem 2. *For any $\alpha\in(0, 1)$ there exists in $\mathfrak{M}$ a distance function ρ^* such that $\rho^*(\phi(x), \phi(y))\leqq\alpha\rho^*(x, y)$.*

Proof. By Theorem 1 there exists a metric $\rho(x, y)$, such that the mapping ϕ is nonexpansive with respect to it. Let

$$\lambda(x, y) = \alpha^{n(x,y)}\rho(x, y).$$

Because $n(\phi(x), \phi(y))=n(x, y)+1$, it follows that

$$\lambda(\phi(x), \phi(y)) \leqq \alpha\lambda(x, y).$$

The function $\lambda(x, y)$ is not in general a metric. However a desired metric $\rho^*(x, y)$ can be defined as

$$\rho^*(x, y) = \inf \sum_{i=1}^{n} \lambda(x_i, x_{i+1})$$

where the infimum is taken over all possible finite systems of elements $x_1, x_2, \cdots, x_{n+1}\in S$ such that $x=x_1$ and $x_{n+1}=y$.

It follows from the definition that $\rho^*(x, y)\leqq\lambda(x, y)\leqq\rho(x, y)$. We will show the validity of the triangle inequality for ρ^*. Let $x, y, z\in S$ and $\epsilon>0$. From the definition of $\rho^*(x, y)$, $\rho^*(y, z)$ there exist elements

$u_1, u_2, \cdots, u_{n+1}, v_1, v_2, \cdots, v_{m+1}$ such that $u_1=x$, $u_{n+1}=y$, $v_1=y$, $v_{m+1}=z$ and

$$\rho^*(x, y) = \sum_{i=1}^{n} \lambda(u_i, u_{i+1}) - \epsilon_1, \qquad \rho^*(y, z) = \sum_{i=1}^{m} \lambda(v_i, v_{i+1}) - \epsilon_2,$$

where $\epsilon_1, \epsilon_2 < \epsilon$. From the definition of $\rho^*(x, z)$ we have

$$\rho^*(x, z) \leqq \sum_{i=1}^{n} \lambda(u_i, u_{i+1}) + \sum_{i=1}^{m} \lambda(v_i, v_{i+1}).$$

Therefore we have

$$\rho^*(x, y) + \rho^*(y, z) \geqq \rho^*(x, z) - \epsilon_1 - \epsilon_2.$$

In order to prove that $\rho^*(x, y)$ is a metric it remains to show that $\rho^*(x, y) \neq 0$ for $x \neq y$. Let $n(x) \leqq n(y)$. From the definition of $\rho^*(x, y)$ it can directly be seen that if $n(x) = n(y)$ then $\rho^*(x, y) \geqq \alpha^{n(x)}\rho(x, y)$ while if $n(x) < n(y)$ then $\rho^*(x, y) \geqq d(x) \cdot \alpha^{n(x)}$ where $d(x)$ is a distance of the point x from the compact set $A_{n(x)+1}$. $d(x)$ is therefore a positive number, hence ρ^* is a distance function. In order to prove that $\rho^* \in \mathfrak{M}$ it remains to show that $x_n \rightarrow_{\rho^*} x$ implies $x_n \rightarrow_{\rho} x$. If this is not the case, then because of compactness with respect to ρ there exists a subsequence, which we denote again by $\{x_n\}$ such that $x_n \rightarrow_{\rho} y$. $y \neq x$, and therefore also that $\rho^*(x_n, y) \rightarrow 0$, which is a contradiction.

It remains to prove that $\rho^*(\phi(x), \phi(y)) \leqq \alpha\rho^*(x, y)$. Let $\epsilon > 0$ be a given number. From the definition of $\rho^*(x, y)$, there exists a representation of $\rho^*(x, y)$ in the form

$$\rho^*(x, y) = \sum_{i=1}^{n} \lambda(x_i, x_{i+1}) - \epsilon_1 \qquad \text{where } \epsilon_1 < \epsilon.$$

Now we have

$$\rho^*(\phi(x), \phi(y)) \leqq \sum_{i=1}^{n} \lambda(\phi(x_i), \phi(x_{i+1})) = \alpha \sum_{i=1}^{n} \lambda(x_i, x_{i+1}),$$

which gives

$$\rho^*(\phi(x), \phi(y)) \leqq \alpha\rho^*(x, y) + \alpha\epsilon_1.$$

Because ϵ was chosen arbitrarily, this proves our theorem.

Reference

1. C. Bessanga, *On the converse of the Banach fixed point principle,* Colloq. Math. 7 (1959), 41–43.

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