MEASURES ON $C(Y)$ WHEN Y IS A COMPACT METRIC SPACE

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1. **Introduction.** Let Y be a compact metric space and $C(Y)$ the space of real-valued continuous functions on Y with the uniform topology. The minimal sigma-algebra containing the closed subsets of $C(Y)$ will be called the Borel subsets of $C(Y)$. If $F(p(1, \cdots, k))$, $p_i \in Y$, (where $p(1, \cdots, k)$ stands for $p_1, \cdots, p_k$) is a consistent family of finite-dimensional Gaussian distributions then sufficient conditions are given to assure the existence of a measure μ on the Borel subsets of $C(Y)$ whose finite-dimensional distributions are those postulated. It will also be shown that such a family of distributions always exists and that Levy's Brownian motion with parameter from the Hilbert space l_2 [1, pp. 293–298] can be taken to have continuous sample paths when the parameter is restricted to certain compact subsets of l_2.

2. The product space $X = \prod_{k=1}^{\infty} [0, 1/2^k]$ is assumed to have the topology induced by the usual l_2 norm $\|\cdot\|$. Let $F(p(1, \cdots, k))$, $p_i \in Y$, be a consistent family of Gaussian distributions. If for some homeomorphism ϕ of Y into X there exists another consistent family of Gaussian distributions $G(q(1, \cdots, k))$, $q_i \in X$, with the property that $F(p(1, \cdots, k)) = G(\phi(p(1, \cdots, k)))$ (where $\phi(p(1, \cdots, k))$ stands for $\phi(p_i), \cdots, \phi(p_k)$ and

$$\iint_{R_2} |s_2 - s_1|^2 dG(q(1, 2))(s_1, s_2) \leqq C\|q_2 - q_1\|^\alpha$$

where $C>0$, $\alpha>0$ are independent of q_1, q_2 in X, then we say the family $F(p(1, \cdots, k))$, $p_i \in Y$, can be extended to X.

THEOREM 1. *If Y is a compact metric space and $F(p(1, \cdots, k))$, $p_i \in Y$, is a consistent family of Gaussian distributions which can be extended to X, then there exists a measure μ on the Borel subsets of $C(Y)$ whose finite-dimensional distributions are those given.*

COROLLARY. *If $Y = \prod_{k=1}^{\infty} [a_k, a_k + 1/2^k]$ where $\{a_k\}$ is in l_2 and*

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$F(p(1, \cdots, k))$, $p_i \in Y$, is the family of Gaussian distributions with mean vector zero and covariance matrix $V=(v_{ij})$ where $2v_{ij}=\|p_i\|+\|p_j\|-\|p_i-p_j\|$ for $i, j=1, \cdots, k$, [1, p. 293], *then there exists a measure μ on $C(Y)$ whose finite-dimensional distributions agree with those given.*

PROOF. Let $\phi((y_1, y_2, \cdots))=(y_1-a_1, y_2-a_2, \cdots)$. Then ϕ is a homeomorphism of Y onto X. Let $G(q(1, \cdots, k))=F(p(1, \cdots, k))$ where $p_i=\phi^{-1}(q_i)$ for $i=1, \cdots, k$. Then

$$\iint_{R_2} | s_2 - s_1|^2 dG(q(1, 2))(s_1, s_2)$$

$$= \iint_{R_2} | t_2 - t_1|^2 dF(p(1, 2))(t_1, t_2) = \|p_1 - p_2\|$$

and the family $F(p(1, \cdots, k))$, $p_i \in Y$, can be extended to X with respect to ϕ. Hence the corollary follows from the previous theorem.

As a result of this corollary we see that Levy's Brownian motion with parameter from the Hilbert space l_2 can be defined to have continuous sample paths on subsets of the form $\prod_{k=1}^{\infty}[a_k, a_k+1/2^k]$. In fact, the conclusions hold if Y is any compact subset of such a set. However, it is known [1, p. 293] that sample function continuity does not hold for arbitrary subsets of l_2.

3. The proof of our theorem will be obtained from the following sequence of lemmas. Throughout the discussion it is assumed that X is as defined above and that $C(X)$ is the space of real-valued continuous functions on X with the uniform topology. Since X is compact $C(X)$ is a complete separable metric space and the results of Prokhorov [2] can be applied.

The Borel sets S of $C(X)$ will be the minimal sigma-algebra containing the closed sets $\mathfrak{F}$ of $C(X)$. The space of all probability measures on the measurable space $(C(X), S)$ will be denoted by $\mathfrak{M}(C(X))$. If $\{m_k\}$ is a sequence of elements in $\mathfrak{M}(C(X))$ we say $\{m_k\}$ converges weakly to m in $\mathfrak{M}(C(X))$ provided that

$$\lim_{n\to\infty} \int_{C(X)} F(f)dm_n = \int_{C(X)} F(f)dm$$

for every bounded continuous functional F on $C(X)$. A metric is defined on $\mathfrak{M}(C(X))$ in the following way. If m_1 and m_2 are in $\mathfrak{M}$ we let $L(m_1, m_2)=\max\{\delta_1, \delta_2\}$ where

$$\delta_1 = \inf\{\epsilon > 0\colon m_1(F) \leqq m_2(F^\epsilon) + \epsilon \quad \text{for all } F \in \mathfrak{F}\},$$

$$\delta_2 = \inf\{\epsilon > 0\colon m_2(F) \leqq m_1(F^\epsilon) + \epsilon \quad \text{for all } F \in \mathfrak{F}\}, \text{ and}$$

$$F^\epsilon = \left\{ f \in C(X): \max_{x\in X} |f(x) - g(x)| < \epsilon \quad \text{for some } g \in F \right\}.$$

In [2, pp. 168–170] it is shown that L-convergence and weak convergence are equivalent in $\mathfrak{M}(C(X))$ and that $\mathfrak{M}(C(X))$ is a complete separable metric space in these topologies. If θ is a continuous mapping of $C(X)$ into $C(X)$ and $m \in \mathfrak{M}(C(X))$ we will denote by m^θ the element of $\mathfrak{M}(C(X))$ such that $m^\theta(A) = m(\theta^{-1}(A))$ for all $A \in S$. If $\delta > 0$ and $m(H) > 1 - \epsilon$, where

$$H = \left\{ f \in C(X): \max_{x\in X} |f(x) - \theta(f)(x)| < \delta \right\},$$

then $L(m, m^\theta) \leqq \max(\epsilon, \delta)$. This estimate is essential to us; its proof is in [2, p. 167].

LEMMA 1. *Let* $H = \prod_{j=1}^N [a_j, b_j]$ *where* $-\infty < a_j < b_j < \infty$ *for* $j = 1, \cdots, N$ *and suppose* $f(x_1, \cdots, x_N)$ *is a real-valued function on* H. *Then there exists a continuous function* $\hat{f}$ *on* H *such that* $\hat{f} = f$ *on* $\mathcal{P} = \{(x_1, \cdots, x_N): x_j = a_j \text{ or } x_j = b_j\}$ *and* $\max_{p\in H} \hat{f}(p) = \max_{p\in\mathcal{P}} \hat{f}(p) \geqq \min_{p\in\mathcal{P}} \hat{f}(p) = \min_{p\in H} \hat{f}(p)$. *In fact, we take* $\hat{f}$ *to be as follows:*

$$\hat{f}(x_1, \cdots, x_N) = \sum_{i=1}^{2^N} f(y_1^i, \cdots, y_N^i) \frac{(x_1 - a_1)^{z_1^i}(b_1 - x_1)^{1-z_1^i} \cdots (x_N - a_N)^{z_N^i}(b_N - x_N)^{1-z_N^i}}{(b_1 - a_1) \cdots (b_N - a_N)}$$

where $\{(y_1^i, \cdots, y_N^i): i = 1, \cdots, 2^N\} = \mathcal{P}$, $z_j^i = 0$ *if* $y_j^i = a_j$, $z_j^i = 1$ *if* $y_j^i = b_j$, *and we assume* $(x_j - a_j)^0 = (b_j - x_j)^0 \equiv 1$.

PROOF. It is clear that $\hat{f}$ is continuous on H and that $\hat{f} = f$ on $\mathcal{P}$. The proof that $\max_{p\in H} \hat{f}(p) = \max_{p\in\mathcal{P}} \hat{f}(p) \geqq \min_{p\in\mathcal{P}} \hat{f}(p) = \min_{p\in H} \hat{f}(p)$ follows by induction on N.

Let $\mathcal{P}_N = \{(x_1, \cdots, x_N, 0, \cdots): x_k = j/2^N, 0 \leqq j \leqq 2^{N-k}\}$ for $N = 1, 2, \cdots$. The number of points of X which $\mathcal{P}_N$ contains will be denoted by $\operatorname{ord}(\mathcal{P}_N)$. Hence we have $\operatorname{ord}(\mathcal{P}_N) \leqq 2^{N(N+1)/2}$. Let

$$H_N = \prod_{k=1}^N [0, 1/2^k], \quad \text{and} \quad H_N^i = \prod_{k=1}^N [a_k^i, a_k^i + 1/2^N]$$

where $a_k^i = j/2^N$ and j is an integer depending on i and k such that $0 \leqq j \leqq 2^{N-k} - 1$ for $k = 1, \cdots, N$. Then there are $2^{N(N-1)/2}$ distinct H_N^i and by relabeling the H_N^i, if necessary, we have H_N as the union of the H_N^i for $1 \leqq i \leqq 2^{N(N-1)/2}$. Moreover, the H_N^i overlap only on their boundaries when they are considered as subsets of R_N. If f is any real-valued function on X we construct the function $\hat{\hat{f}}_N$ on X such that $\hat{\hat{f}}_N(x_1, \cdots, x_N, x_{N+1}, \cdots) = \hat{f}_i(x_i, \cdots, x_N, 0, 0, \cdots)$ for $(x_1, \cdots, x_N, 0, \cdots)$ in $H_N^i \times (0, 0, \cdots)$ and where $\hat{f}_i(x_1, \cdots, x_N, 0, \cdots)$ is defined on $H_N^i \times (0, 0, \cdots)$ by using Lemma 1 on $f(x_1, \cdots, x_N, 0, \cdots)$ and the obvious homeomorphism. Then $\hat{\hat{f}}_N$ is continuous on X and $\hat{\hat{f}}_N = f$ on $\mathcal{P}_N$. Moreover, the maximum and minimum of $\hat{\hat{f}}_N$ over X are both attained on $\mathcal{P}_N$.

Definition. If f is in $C(X)$ let $G_N(f) = \hat{\hat{f}}_N$ where $\hat{\hat{f}}_N$ is defined as above. Let $S_N = G_N(C(X))$ for $N = 1, 2, \cdots$.

From the above considerations it is quite clear that G_N is a continuous mapping of $C(X)$ onto the closed linear subspace S_N of $C(X)$ and that S_N is homeomorphic to R_L where $L = \operatorname{ord}(\mathcal{P}_N)$. Let $G(q(1, \cdots, k))$, $q_i \in X$, be a system of consistent Gaussian distributions. If E is any Borel subset of R_L and $I = \{f \in S_N: [f(q_1), \cdots, f(q_L)] \in E\}$ where $\{q_1, \cdots, q_L\} = \mathcal{P}_N$ then we define $m_N(I) = \int_E dG(q(1, \cdots, L))$. Then m_N is a measure on the Borel subsets of S_N and since S_N is a closed subset of $C(X)$ we can extend m_N to a measure on the Borel subsets of $C(X)$ simply by letting $m_N(C(X) - S_N) = 0$. That is, m_N so extended is an element of $\mathfrak{M}(C(X))$. Moreover, if $\theta = G_{N-1}$ then $m_N^\theta = m_{N-1}$ for $N = 2, 3, \cdots$.

Lemma 2. *The sequence $\{m_N\}$ is a Cauchy sequence in $\mathfrak{M}(C(X))$ provided*

$$\iint_{R_2} |s_2 - s_1|^2 dG(q(1, 2))(s_1, s_2) \leqq C \|q_1 - q_2\|^\alpha$$

for all q_1, q_2 in X where $C > 0$ and $\alpha > 0$ are constants independent of q_1, q_2 and $\|q_1 - q_2\|$ is the usual l_2 distance between q_1, q_2.

Proof. We will examine $L(m_N, m_{N-1})$ by noting that $L(m_N, m_{N-1}) = L(m_N, m_N^\theta)$ where $\theta = G_{N-1}$. Since $m_N(C(X) - S_N) = 0$ we have

$$\begin{aligned} J &= m_N\left\{f \in C(X):\ \max_{q \in X} |\theta(f)(q) - f(q)| > 1/N^2\right\} \\ &= m_N\left\{f \in S_N:\ \max_{q \in X} |\theta(f)(q) - f(q)| > 1/N^2\right\}. \end{aligned}$$

Now if $f \in S_N$ then $\theta(f)$ is in S_{N-1} and

$$\max_{q \in X} |f(q) - \theta(f)(q)| = \max_{1 \leqq i \leqq K} \left[\max_{q \in Z_i} |f(q) - \theta(f)(q)|\right]$$

where $K = 2^{(N-1)(N-2)/2}$ and $Z_i = H_{N-1}^i \times [0, 1/2^N] \times (0, \cdots)$. Furthermore, if $f \in S_N$ Lemma 1 implies that

$$\max_{q\in Z_i} |f(q)-\theta(f)(q)| \leqq |f(q_1)-f(q_2)|$$

where $q_1 \in \mathcal{P}_N \cap Z_i$, $q_2 \in \mathcal{P}_{N-1} \cap Z_i$, and $q_1 \neq q_2$ depend on f. Hence

$$J \leqq \sum_{i=1}^{K} m_N \left\{ f\in S_N: \max_{q\in Z_i} |f(q)-\theta(f)(q)| > 1/N^2 \right\}$$

and we see that

$$m_N\left\{f\in S_N: \max_{q\in Z_i} |f(q)-\theta(f)(q)| > 1/N^2\right\} \leqq \sum_{(q_1,q_2)\in A} m_N\{f\in S_N: |f(q_1)-f(q_2)| > 1/N^2\}$$

where $A=\{(q_1,q_2): q_1\in\mathcal{P}_N\cap Z_i,\ q_2\in\mathcal{P}_{N-1}\cap Z_i,\ q_1\neq q_2\}$. Now $(q_1,q_2)\in A$ implies $\|q_1-q_2\|<N^{1/2}/2^{N-1}$ so for $(q_1,q_2)\in A$ and γ a positive integer we have

$$\begin{aligned} m_N\{f\in S_N: |f(q_1)-f(q_2)| > 1/N^2\} &= m_N\{f\in S_N: |f(q_1)-f(q_2)|^{2N\cdot\gamma} > (1/N^2)^{2N\gamma}\} \\ &\leqq N^{4N\gamma}\int_{S_N} |f(q_1)-f(q_2)|^{2N\gamma} dm_N \\ &= N^{4N\gamma}[\mathrm{Var}(f(q_1)-f(q_2))]^{N\gamma}[1\cdot 3\cdot 5 \cdots (2N\gamma-1)] \\ &\leqq CN^{4N\gamma}[N^{1/2}/2^{N-1}]^{N\gamma\alpha}[1\cdot 3\cdot 5\cdots(2N\gamma-1)]. \end{aligned}$$

Letting $\mathrm{ord}(A)$ denote the number of elements of A we have $\mathrm{ord}(A) \leqq 3^N\cdot 2^N$ and $K\cdot\mathrm{ord}(A)\leqq 2^{N^2/2+3N/2+1}$. Hence

$$J \leqq C\cdot 2^{N^2/2+3N/2+1}\cdot N^{4N\gamma}[1\cdot 3\cdot 5\cdots(2N\gamma-1)][N^{1/2}/2^{N-1}]^{N\gamma\alpha}$$

and choosing γ so that $\gamma\cdot\alpha\geqq 1$ we see that $J\leqq \exp\{-N\gamma\}$ for N sufficiently large. As a result of the estimate [2, p. 167] we have $L(m_N, m_{N-1})\leqq\max\{e^{-N\gamma}, N^{-2}\}$ for N sufficiently large and hence $\{m_N\}$ is a Cauchy sequence in the L-metric so our proof is complete.

If $G(q(1,\cdots,k))$, $q_i\in X$, is a consistent family of Gaussian distributions satisfying the conditions of Lemma 2 then $\{m_N\}$ is a Cauchy sequence in $\mathfrak{M}(C(X))$ and there exists an element m in $\mathfrak{M}C(X)$ such that $\{m_N\}$ converges weakly to m. Moreover, $m_N{}^{G_M}$ will converge weakly to m^{G_M} and since $m_N{}^{G_M}=m_M$ for $N\geqq M$ we have $m^{G_M}=m_M$. That is, if $\{q_1,\cdots,q_k\}\subseteq\mathcal{P}_N$ for some N then

$$m\{f\in C(X): [f(q_1),\cdots,f(q_k)]\in E\} = \int_E dG(q(1,\cdots,k))$$

for any Borel subset E of R_k. To show that all the finite-dimensional distributions of m coincide with those postulated follows quite easily.

The following theorem summarizes the results of this section.

THEOREM 2. *If $G(q(1, \cdots, k))$, $q_i \in X$, is a consistent family of Gaussian distributions then there exists a measure m on the Borel subsets of $C(X)$ whose finite-dimensional distributions coincide with those postulated provided that for all q_1, q_2 in X*

$$\int\int_{R_2} | s_2 - s_1 |^2 dG(q(1, 2))(s_1, s_2) \leqq C \| q_1 - q_2 \|^\alpha$$

where $C > 0$ and $\alpha > 0$ are independent of q_1, q_2.

4. Proof of Theorem 1. Since the family $F(p(1, \cdots, k))$, $p_i \in Y$, can be extended to X there exists a consistent family of Gaussian distributions $G(q(1, \cdots, k))$, $q_i \in X$, which satisfies the conditions of Theorem 2 and $F(p(1, \cdots, k)) = G(\phi(p(1, \cdots, k)))$ where ϕ is a homeomorphism of Y into X. Let m be the measure on $C(X)$ obtained as a result of Theorem 2. Since ϕ is a homeomorphism of Y into X we have $\theta(f) = f(\phi(\cdot))$ mapping $C(X)$ continuously onto $C(Y)$. Then m^θ is a measure on the Borel subsets of $C(Y)$ and if $I = \{g \in C(Y): [g(p_1), \cdots, g(p_k)] \in E\}$ where $p_1, \cdots, p_k$ are in Y and E is a Borel subset of R_k we have

$$\begin{aligned} m^\theta(I) &= m(\theta^{-1}(I)) \\ &= m\{f \in C(X): [\theta(f)(p_1), \cdots, \theta(f)(p_k)] \in E\} \\ &= \int_E dG(\phi(p(1, \cdots, k))) = \int_E dF(p(1, \cdots, k)). \end{aligned}$$

Hence $\mu = m^\theta$ has the finite-dimensional distributions postulated and the theorem is proved.

5. Since Y is a compact metric space there exists a homeomorphism ϕ of Y into X. Consequently, if $G(q(1, \cdots, k))$, $q_i \in X$, is the family of distributions mentioned in the proof of the Corollary to Theorem 1 then the family $F(p(1, \cdots, k)) = G(\phi(p(1, \cdots, k)))$, $p_i \in Y$, satisfies Theorem 1 so the existence question mentioned in the introduction is easily handled.

As a final remark we mention that the conclusions of Theorem 2 hold if the hypotheses are such that the integral

$$\int\int_{R_2} | s_2 - s_1 |^2 dG(q(1, 2))(s_1, s_2)$$

is less than or equal to $C|q_1-q_2|^\alpha$ where $C>0$, $\alpha>0$ are independent of q_1, q_2 and $|q_1-q_2|=\sum_{i=1}^{\infty}|b_i-a_i|$ when $q_1=(a_1, a_2, \cdots)$, $q_2=(b_1, b_2, \cdots)$.

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