MEASURES ON $C(Y)$ WHEN $Y$ IS A COMPACT METRIC SPACE

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1. Introduction. Let $Y$ be a compact metric space and $C(Y)$ the space of real-valued continuous functions on $Y$ with the uniform topology. The minimal sigma-algebra containing the closed subsets of $C(Y)$ will be called the Borel subsets of $C(Y)$. If $F(p(1, \cdots, k)), p_i \in Y$, (where $p(1, \cdots, k)$ stands for $p_1, \cdots, p_k$) is a consistent family of finite-dimensional Gaussian distributions then sufficient conditions are given to assure the existence of a measure $\mu$ on the Borel subsets of $C(Y)$ whose finite-dimensional distributions are those postulated. It will also be shown that such a family of distributions always exists and that Levy’s Brownian motion with parameter from the Hilbert space $l_2$ [1, pp. 293–298] can be taken to have continuous sample paths when the parameter is restricted to certain compact subsets of $l_2$.

2. The product space $X = \prod_{k=1}^{\infty} [0, 1/2^k]$ is assumed to have the topology induced by the usual $l_2$ norm $\| \cdot \|$. Let $F(p(1, \cdots, k)), p_i \in Y$, be a consistent family of Gaussian distributions. If for some homeomorphism $\varphi$ of $Y$ into $X$ there exists another consistent family of Gaussian distributions $G(q(1, \cdots, k)), q_i \in X$, with the property that $F(p(1, \cdots, k)) = G(\varphi(p(1, \cdots, k)))$ (where $\varphi(p(1, \cdots, k))$ stands for $\varphi(p_1), \cdots, \varphi(p_k)$ and

$$\int \int_{R^2} \left| s_2 - s_1 \right|^2 dG(q(1, 2))(s_1, s_2) \leq C \| q_2 - q_1 \|^\alpha$$

where $C > 0$, $\alpha > 0$ are independent of $q_1, q_2$ in $X$, then we say the family $F(p(1, \cdots, k)), p_i \in Y$, can be extended to $X$.

Theorem 1. If $Y$ is a compact metric space and $F(p(1, \cdots, k)), p_i \in Y$, is a consistent family of Gaussian distributions which can be extended to $X$, then there exists a measure $\mu$ on the Borel subsets of $C(Y)$ whose finite-dimensional distributions are those given.

Corollary. If $Y = \prod_{k=1}^{\infty} [a_k, a_k + 1/2^k]$ where $\{a_k\}$ is in $l_2$ and

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$F(p(1, \cdots, k)), \ p \in Y$, is the family of Gaussian distributions with mean vector zero and covariance matrix $V = (v_{ij})$ where

$$v_{ij} = \|p_i\| + \|p_j\|, \ \text{for} \ i, j = 1, \cdots, k, \ [1, \text{p. 293}],$$

then there exists a measure $\mu$ on $C(Y)$ whose finite-dimensional distributions agree with those given.

**Proof.** Let $\phi((y_1, y_2, \cdots)) = (y_1 - a_1, y_2 - a_2, \cdots)$. Then $\phi$ is a homeomorphism of $Y$ onto $X$. Let $G(q(1, \cdots, k)) = F(p(1, \cdots, k))$ where $p_i = \phi^{-1}(q_i)$ for $i = 1, \cdots, k$. Then

$$\int \int_{R^2} |s_2 - s_1|^2 dG(q(1, 2))(s_1, s_2)
= \int \int_{R^2} |t_2 - t_1|^2 dF(p(1, 2))(t_1, t_2) = \|p_1 - p_2\|
$$

and the family $F(p(1, \cdots, k)), \ p \in Y$, can be extended to $X$ with respect to $\phi$. Hence the corollary follows from the previous theorem.

As a result of this corollary we see that Levy's Brownian motion with parameter from the Hilbert space $l_2$ can be defined to have continuous sample paths on subsets of the form $\prod_{k=1}^{\infty} [a_k, a_k + 1/2^k]$. In fact, the conclusions hold if $Y$ is any compact subset of such a set. However, it is known [1, p. 293] that sample function continuity does not hold for arbitrary subsets of $l_2$.

3. The proof of our theorem will be obtained from the following sequence of lemmas. Throughout the discussion it is assumed that $X$ is as defined above and that $C(X)$ is the space of real-valued continuous functions on $X$ with the uniform topology. Since $X$ is compact $C(X)$ is a complete separable metric space and the results of Prokhorov [2] can be applied.

The Borel sets $S$ of $C(X)$ will be the minimal sigma-algebra containing the closed sets $\mathcal{F}$ of $C(X)$. The space of all probability measures on the measurable space $(C(X), S)$ will be denoted by $\mathfrak{M}(C(X))$. If $\{m_k\}$ is a sequence of elements in $\mathfrak{M}(C(X))$ we say $\{m_k\}$ converges weakly to $m$ in $\mathfrak{M}(C(X))$ provided that

$$\lim_{n\to\infty} \int_{C(X)} F(f) dm_n = \int_{C(X)} F(f) dm$$

for every bounded continuous functional $F$ on $C(X)$. A metric is defined on $\mathfrak{M}(C(X))$ in the following way. If $m_1$ and $m_2$ are in $\mathfrak{M}$ we let $L(m_1, m_2) = \max\{\delta_1, \delta_2\}$ where

$$\delta_1 = \inf\{\epsilon > 0: m_1(F) \leq m_2(F^\epsilon) + \epsilon \ \text{for all} \ F \in \mathcal{F}\},$$

$$\delta_2 = \inf\{\epsilon > 0: m_2(F) \leq m_1(F^\epsilon) + \epsilon \ \text{for all} \ F \in \mathcal{F}\},$$

and
\[ F^* = \{ f \in C(X) : \max_{x \in X} |f(x) - g(x)| < \epsilon \text{ for some } g \in F \} . \]

In [2, pp. 168–170] it is shown that \( L \)-convergence and weak convergence are equivalent in \( S_\ell(C(X)) \) and that \( S_\ell(C(X)) \) is a complete separable metric space in these topologies. If \( \theta \) is a continuous mapping of \( C(X) \) into \( C(Y) \) and \( m \in S_\ell(C(X)) \) we will denote by \( m^\theta \) the element of \( S_\ell(C(X)) \) such that \( m^\theta(A) = m(\theta^{-1}(A)) \) for all \( A \in S \).

If \( \delta > 0 \) and \( m(H) > 1 - \varepsilon \), where
\[
H = \{ f \in C(X) : \max_{x \in X} |f(x) - \theta(f)(x)| < \delta \},
\]
then \( L(m, m^\theta) \leq \max(\varepsilon, \delta) \). This estimate is essential to us; its proof is in [2, p. 167].

**Lemma 1.** Let \( H = \prod_{j=1}^{N} [a_j, b_j] \) where \( -\infty < a_j < b_j < \infty \) for \( j = 1, \ldots, N \) and suppose \( f(x_1, \ldots, x_N) \) is a real-valued function on \( H \). Then there exists a continuous function \( \hat{f} \) on \( H \) such that \( \hat{f} = f \) on \( \mathcal{P} = \{ (x_1, \ldots, x_N) : x_j = a_j \text{ or } x_j = b_j \} \) and \( \max_{p \in H} \hat{f}(p) = \max_{p \in \mathcal{P}} f(p) \geq \min_{p \in \mathcal{P}} f(p) = \min_{p \in H} \hat{f}(p) \). In fact, we take \( \hat{f} \) to be as follows:

\[
\hat{f}(x_1, \ldots, x_N) = \sum_{i=1}^{2^N} f(y_i^1, \ldots, y_i^N) \cdot \frac{(x_1 - a_1)^{i_1}(b_1 - x_1)^{1-i_1} \cdots (x_N - a_N)^{i_N}(b_N - x_N)^{1-i_N}}{(b_1 - a_1) \cdots (b_N - a_N)}
\]

where \( \{ y_i^1, \ldots, y_i^N \} : i = 1, \ldots, 2^N \} = \mathcal{P} \), \( a_j^i = 0 \) if \( y_j^i = a_j \), \( a_j^i = 1 \) if \( y_j^i = b_j \), and we assume \( (x_j - a_j)^0 = (b_j - x_j)^0 = 1 \).

**Proof.** It is clear that \( \hat{f} \) is continuous on \( H \) and that \( \hat{f} = f \) on \( \mathcal{P} \). The proof that \( \max_{p \in H} \hat{f}(p) = \max_{p \in \mathcal{P}} f(p) = \min_{p \in \mathcal{P}} f(p) = \min_{p \in H} \hat{f}(p) \) follows by induction on \( N \).

Let \( \mathcal{P}_N = \{ (x_1, \ldots, x_N, 0, \ldots) : x_k = j/2^N, \ 0 \leq j \leq 2^N - k \} \) for \( N = 1, 2, \ldots \). The number of points of \( X \) which \( \mathcal{P}_N \) contains will be denoted by \( \text{ord}(\mathcal{P}_N) \). Hence we have \( \text{ord}(\mathcal{P}_N) \leq 2^{N(N+1)/2} \).

Let \( H_N = \prod_{k=1}^{N} [0, 1/2^k], \) and \( H^i_N = \prod_{k=1}^{N} [a^i_k, a^i_k + 1/2^N] \)

where \( a^i_k = j/2^N \) and \( j \) is an integer depending on \( i \) and \( k \) such that \( 0 \leq j \leq 2^N - k - 1 \) for \( k = 1, \ldots, N \). Then there are \( 2^{N(N-1)/2} \) distinct \( H_N^i \) and by relabeling the \( H_N^i \), if necessary, we have \( H_N \) as the union of the \( H_N^i \) for \( 1 \leq i \leq 2^{N(N-1)/2} \). Moreover, the \( H_N^i \) overlap only on
their boundaries when they are considered as subsets of $R_N$. If $f$ is any real-valued function on $X$ we construct the function $\hat{f}_N$ on $X$ such that $\hat{f}_N(x_1, \ldots, x_N, x_{N+1}, \ldots) = \hat{f}_i(x_i, \ldots, x_N, 0, 0, \ldots)$ for $(x_1, \ldots, x_N, 0, 0, \ldots)$ in $H_N \times (0, 0, \ldots)$ and where $\hat{f}_i(x_1, \ldots, x_N, 0, 0, \ldots)$ is defined on $H_i \times (0, 0, \ldots)$ by using Lemma 1 on $f(x_1, \ldots, x_N, 0, 0, \ldots)$ and the obvious homeomorphism. Then $\hat{f}_N$ is continuous on $X$ and $\hat{f}_N = f$ on $\partial \mathcal{N}$. Moreover, the maximum and minimum of $\hat{f}_N$ over $X$ are both attained on $\partial \mathcal{N}$.

**Definition.** If $f$ is in $C(X)$ let $G_N(f) = \hat{f}_N$ where $\hat{f}_N$ is defined as above. Let $S_N = G_N(C(X))$ for $N = 1, 2, \ldots$.

From the above considerations it is quite clear that $G_N$ is a continuous mapping of $C(X)$ onto the closed linear subspace $S_N$ of $C(X)$ and that $S_N$ is homeomorphic to $R_L$ where $L = \text{ord}(\mathcal{N})$. Let $G(q(1, \ldots, k)), q_i \in X$, be a system of consistent Gaussian distributions. If $E$ is any Borel subset of $R_L$ and $I = \{ f \in S_N : [f(q_1), \ldots, f(q_L)] \in E \}$ where $\{q_1, \ldots, q_L\} = \mathcal{N}$ then we define $m_N(I) = \int_E dG(q(1, \ldots, L))$. Then $m_N$ is a measure on the Borel subsets of $S_N$ and since $S_N$ is a closed subset of $C(X)$ we can extend $m_N$ to a measure on the Borel subsets of $C(X)$ simply by letting $m_N(C(X) - S_N) = 0$. That is, $m_N$ so extended is an element of $\mathfrak{M}(C(X))$. Moreover, if $\theta = G_{N-1}$ then $m_N^\theta = m_{N-1}$ for $N = 2, 3, \ldots$.

**Lemma 2.** The sequence $\{m_N\}$ is a Cauchy sequence in $\mathfrak{M}(C(X))$ provided

$$\int \int_{R^2_2} |s_2 - s_1|^2 dG(q(1, 2))(s_1, s_2) \leq C ||q_1 - q_2||^\alpha$$

for all $q_1, q_2$ in $X$ where $C > 0$ and $\alpha > 0$ are constants independent of $q_1, q_2$ and $||q_1 - q_2||$ is the usual $l_2$ distance between $q_1, q_2$.

**Proof.** We will examine $L(m_N, m_{N-1})$ by noting that $L(m_N, m_{N-1}) = L(m_N, m_N^\theta)$ where $\theta = G_{N-1}$. Since $m_N(C(X) - S_N) = 0$ we have

$$J = m_N \left\{ f \in C(X) : \max_{q \in X} | \theta(f)(q) - f(q) | > 1/N^2 \right\}$$

$$= m_N \left\{ f \in S_N : \max_{q \in X} | \theta(f)(q) - f(q) | > 1/N^2 \right\}.$$ 

Now if $f \in S_N$ then $\theta(f)$ is in $S_{N-1}$ and

$$\max_{q \in X} | f(q) - \theta(f)(q) | = \max_{1 \leq i \leq K} \left[ \max_{q \in Z_i} | f(q) - \theta(f)(q) | \right]$$

where $K = 2^{(N-1)(N-2)/2}$ and $Z_i = H_{N-1} \times [0, 1/2^N] \times (0, \ldots)$. Furthermore, if $f \in S_N$ Lemma 1 implies that
where $q_1 \in \Theta_N \cap Z_i$, $q_2 \in \Theta_{N-1} \cap Z_i$, and $q_1 \neq q_2$ depend on $f$. Hence

$$J \leq \sum_{i=1}^{K} m_N \{ f \in S_N : \max_{q \in Z_i} |f(q) - \theta(f)(q)| > 1/N^2 \}$$

and we see that

$$m_N \{ f \in S_N : \max_{q \in Z_i} |f(q) - \theta(f)(q)| > 1/N^2 \}$$

where $A = \{(q_1, q_2) : q_1 \in \Theta_N \cap Z_i, q_2 \in \Theta_{N-1} \cap Z_i, q_1 \neq q_2 \}$. Now $(q_1, q_2) \in A$ implies $|q_1 - q_2| < N^{1/2}/2^{N-1}$ so for $(q_1, q_2) \in A$ and $\gamma$ a positive integer we have

$$m_N \{ f \in S_N : |f(q_1) - f(q_2)| > 1/N^2 \} = m_N \{ f \in S_N : |f(q_1) - f(q_2)|^{2N^\gamma} > (1/N^2)^{2N^\gamma} \} \leq N^{4N^\gamma} \int_{S_N} |f(q_1) - f(q_2)|^{2N^\gamma} dm_N$$

$$= N^{4N^\gamma} \left[ \text{Var}(f(q_1) - f(q_2)) \right]^{N^\gamma} \left[ 1 \cdot 3 \cdot 5 \cdots (2N^\gamma - 1) \right] \leq CN^{4N^\gamma} \left[ N^{1/2}/2^{N-1} \right]^{N^\gamma} \left[ 1 \cdot 3 \cdot 5 \cdots (2N^\gamma - 1) \right].$$

Letting $\text{ord}(A)$ denote the number of elements of $A$ we have $\text{ord}(A) \leq 3N \cdot 2N$ and $K \cdot \text{ord}(A) \leq 2N^{3/2 + 3N/2 + 1}$. Hence

$$J \leq C \cdot 2N^{3/2 + 3N/2 + 1} \cdot N^{4N^\gamma} \left[ 1 \cdot 3 \cdot 5 \cdots (2N^\gamma - 1) \right] \left[ N^{1/2}/2^{N-1} \right]^{N^\gamma}$$

and choosing $\gamma$ so that $\gamma \cdot \alpha \geq 1$ we see that $J \leq \exp \{ -N^\gamma \}$ for $N$ sufficiently large. As a result of the estimate [2, p. 167] we have $L(m_N, m_{N-1}) \leq \max \{ e^{-N^\gamma}, N^{-2} \}$ for $N$ sufficiently large and hence $\{ m_N \}$ is a Cauchy sequence in the $L$-metric so our proof is complete.

If $G(q(1, \cdots, k)), q_i \in X$, is a consistent family of Gaussian distributions satisfying the conditions of Lemma 2 then $\{ m_N \}$ is a Cauchy sequence in $\mathcal{M}(C(X))$ and there exists an element $m$ in $\mathcal{M}(C(X))$ such that $\{ m_N \}$ converges weakly to $m$. Moreover, $m_N^{GM}$ will converge weakly to $m^{GM}$ and since $m_N^{GM} = m_{GM}$ for $N \geq M$ we have $m^{GM} = m_{GM}$. That is, if $\{ q_1, \cdots, q_k \} \subseteq \Theta_N$ for some $N$ then

$m \{ f \in C(X) : [f(q_1), \cdots, f(q_k)] \in E \} = \int_E dG(q(1, \cdots, k))$
for any Borel subset $E$ of $R_k$. To show that all the finite-dimensional distributions of $m$ coincide with those postulated follows quite easily.

The following theorem summarizes the results of this section.

**Theorem 2.** If $G(q(1, \cdots, k))$, $q_i \in X$, is a consistent family of Gaussian distributions then there exists a measure $m$ on the Borel subsets of $C(X)$ whose finite-dimensional distributions coincide with those postulated provided that for all $q_1, q_2$ in $X$

$$\int \int_{R_2} |s_2 - s_1|^2 dG(q(1, 2))(s_1, s_2) \leq C||q_1 - q_2||^\alpha$$

where $C > 0$ and $\alpha > 0$ are independent of $q_1, q_2$.

**4. Proof of Theorem 1.** Since the family $F(p(1, \cdots, k))$, $p_i \in Y$, can be extended to $X$ there exists a consistent family of Gaussian distributions $G(q(1, \cdots, k))$, $q_i \in X$, which satisfies the conditions of Theorem 2 and $F(p(1, \cdots, k)) = G(\phi(p(1, \cdots, k)))$ where $\phi$ is a homeomorphism of $Y$ into $X$. Let $m$ be the measure on $C(X)$ obtained as a result of Theorem 2. Since $\phi$ is a homeomorphism of $Y$ into $X$ we have $\theta(f) = f(\phi(\cdot))$ mapping $C(X)$ continuously onto $C(Y)$. Then $m^\theta$ is a measure on the Borel subsets of $C(Y)$ and if $I = \{g \in C(Y) \in C(X): \theta(p_1), \cdots, \theta(p_k) \in E\}$ where $p_1, \cdots, p_k$ are in $Y$ and $E$ is a Borel subset of $R_k$ we have

$$m^\theta(I) = m(\theta^{-1}(I))$$

$$= m\{f \in C(X): \theta(f)(p_1), \cdots, \theta(f)(p_k) \in E\}$$

$$= \int_E dG(\phi(p(1, \cdots, k))) = \int_E dF(p(1, \cdots, k)).$$

Hence $\mu = m^\theta$ has the finite-dimensional distributions postulated and the theorem is proved.

**5.** Since $Y$ is a compact metric space there exists a homeomorphism $\phi$ of $Y$ into $X$. Consequently, if $G(q(1, \cdots, k))$, $q_i \in X$, is the family of distributions mentioned in the proof of the Corollary to Theorem 1 then the family $F(p(1, \cdots, k)) = G(\phi(p(1, \cdots, k)))$, $p_i \in Y$, satisfies Theorem 1 so the existence question mentioned in the introduction is easily handled.

As a final remark we mention that the conclusions of Theorem 2 hold if the hypotheses are such that the integral

$$\int \int_{R_2} |s_2 - s_1|^2 dG(q(1, 2))(s_1, s_2)$$
is less than or equal to $C|q_1 - q_2|^{\alpha}$ where $C > 0$, $\alpha > 0$ are independent of $q_1$, $q_2$ and $|q_1 - q_2| = \sum_{i=1}^{\infty} |b_i - a_i|$ when $q_1 = (a_1, a_2, \ldots)$, $q_2 = (b_1, b_2, \ldots)$.

**Bibliography**


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