LATTICE POINTS IN REGIONS

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1. Let S be a bounded set of points in the Euclidean plane with a unit distance defined. If a rectangular coordinate system is imposed, a certain number of points of S are lattice points, i.e. points with integer coordinates. Let $m(S)$ be the minimum number of lattice points of S under all possible choices of the axis system, and $M(S)$ the maximum number. For example if S is a closed disk of diameter one, then $m(S)=0$ and $M(S)=2$. The definitions of $m(S)$ and $M(S)$ could be given in terms of a fixed rectangular coordinate system, with the set S being freely rotated and translated in the plane. It will be convenient in the proofs to use sometimes one and sometimes the other of these two formulations.

Although the definitions and theorems of this paper are given for 2-dimensional Euclidean space, the generalization to higher dimensions involves no difficulties whatsoever.

It is apparent that $m(S) \leqq M(S)$ for any set S. R. M. Robinson suggested that the strict inequality holds for a nonempty bounded closed set, which is a more general result than we had formulated.

THEOREM 1. *If S is a nonempty bounded closed set then $m(S) < M(S)$.*

PROOF. Select any points A and B with $A \in S$ and $B \notin S$. On the straight line segment AB let P be the point of S that is closest to B; P exists because S is closed. Now impose a coordinate system with origin at P. With this coordinate system suppose that S has r lattice points, so that $m(S) \leqq r \leqq M(S)$. Let $Q_1, Q_2, \cdots, Q_k$ be all the lattice points in the plane which are not members of S and each of which is within unit distance of some point of S; this is a finite set of points because S is bounded. For each of these points Q_i there is a positive number δ_i so that the disk of radius δ_i with center at Q_i contains no point of S. Define δ as $\min(\delta_1, \delta_2, \cdots, \delta_k)$.

With the set S held fixed, translate the coordinate system a distance $\delta/2$ in the direction from P to B. One lattice point at least is thereby removed from S, and no lattice point is gained. With this new coordinate system, the set S has at most $r-1$ lattice points and so

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$$m(S) \leqq r-1 < r \leqq M(S).$$

2. **Measurable regions.** We turn now to a much more restricted class of point sets and establish the following result.

THEOREM 2. *If R is a bounded measurable region then $m(R) \leqq \mu(R) \leqq M(R)$, where $\mu(R)$ denotes the measure of R.*

If $\mu(R)$ is not an integer the inequalities can of course be made strict.

PROOF. Relative to any fixed coordinate system define the function

$$\begin{aligned} \phi(x, y) &= 1 \qquad \text{if } (x, y) \in R, \\ &= 0 \qquad \text{if } (x, y) \notin R. \end{aligned}$$

Define $\psi(x, y)$ by

$$\begin{aligned} \psi(x, y) &= 1 \qquad \text{if } 0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1, \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

If i and j are any integers then

$$\begin{aligned} \int_0^1 \int_0^1 \phi(x+i, y+j)dxdy &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x+i, y+j)\psi(x, y)dxdy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x, y)\psi(x-i, y-j)dxdy \qquad (1) \\ &= \iint_{R \cap Q_{ij}} dxdy \end{aligned}$$

where Q_{ij} is the unit square $i \leqq x \leqq i+1$, $j \leqq y \leqq j+1$. Next define

$$C(R, x, y) = \sum_{i,j} \phi(x+i, y+j)$$

where the sum extends over all pairs of integers i, j. The sum is finite since R is bounded. Thus $C(R, x, y)$ counts the number of points $(x+i, y+j)$, $i=0, \pm 1, \pm 2, \cdots, j=0, \pm 1, \pm 2, \cdots$, that belong to R. Thus $C(R, 0, 0)$ is the number of lattice points belonging to R. If we think of a new coordinate system parallel to the old one but with origin at (x, y) we see that $C(R, x, y)$ counts the number of lattice points belonging to R relative to the new coordinate system. Therefore we have

$$m(R) \leqq C(R, x, y) \leqq M(R) \qquad (2)$$

for every pair of real numbers x, y. Also by (1) we see that

$$\begin{aligned}\int_0^1\int_0^1 C(R,x,y)dxdy &= \sum_{i,j}\int_0^1\int_0^1 \phi(x+i,y+j)dxdy \\ &= \sum_{i,j}\iint_{R\cap\Omega_{ij}} dxdy = \iint_R dxdy = \mu(R).\end{aligned} \tag{3}$$

This with (2) implies that $m(R) \leqq \mu(R) \leqq M(R)$ because, for example,

$$\int_0^1\int_0^1 m(R)dxdy = m(R).$$

If we also assume that R is closed and nonempty, Theorem 2 can be strengthened.

THEOREM 3. *If R is a closed, nonempty, bounded, measurable region then* $m(R) \leqq \mu(R) < M(R)$.

PROOF. Impose a coordinate system so that R covers exactly $m(R)$ lattice points, so that $m(R) = C(R, 0, 0)$ in the notation of the proof of Theorem 2. Let $\delta < 1$ be a positive number so that for each lattice point P not in R the distance from P to every point of R exceeds δ. It follows that $C(R, x, y) = m(R)$ for all (x, y) satisfying $x^2+y^2 \leqq \delta^2$. Let T_1 be the region defined by

$$0 \leqq x, \quad 0 \leqq y, \qquad x^2 + y^2 \leqq \delta^2, \tag{4}$$

and T_2 the region

$$0 \leqq x \leqq 1, \quad 0 \leqq y \leqq 1, \qquad x^2 + y^2 > \delta^2. \tag{5}$$

First we observe that

$$\iint_{T_1} C(R, x, y)dxdy = \iint_{T_1} m(R)dxdy.$$

Also, since the two regions T_1 and T_2 comprise the unit square, $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, equation (3) gives

$$\begin{aligned}\mu(R) &= \iint_{T_1} C(R,x,y)dxdy + \iint_{T_2} C(R,x,y)dxdy \\ &\leqq \iint_{T_1} m(R)dxdy + \iint_{T_2} M(R)dxdy \\ &= \pi\delta^2 m(R)/4 + M(R)(1 - \pi\delta^2/4) \\ &= M(R) - \{M(R) - m(R)\}\pi\delta^2/4 < M(R)\end{aligned}$$

by Theorem 1.

3. **Regions constructed of disks.** The inequalities of Theorems 2 and 3 can be made strict if we restrict the class of regions as follows. We begin with a measurable topological disk D, i.e., the topological equivalent of a circle plus interior. Let D_1 be a measurable topological disk such that $D_1 \subset D$ and such that the intersection of the boundaries of D_1 and D is at most a finite set of points.

Let T_1 be the closure of $D - D_1$. Let D_2 be a measurable topological disk such that $D_2 \subset T_1$ and such that the intersection of the boundaries of D_2 and T_1 is at most a finite set of points. Let T_2 be the closure of $T_1 - D_2$. Continuing, let D_3 be a measurable topological disk such that $D_3 \subset T_2$ and such that the intersection of the boundaries of D_3 and T_2 is at most a finite set of points. Let T_3 be the closure of $T_2 - D_3$. Repeat this procedure a finite number of times, obtaining the closed set T_j. Say that an *admissible* region is one obtained by taking the union of a finite number of disjoint nonempty sets of the type T_j.

We note that an admissible region satisfies the hypotheses of Theorem 3. Also, an admissible region has a well-defined boundary, namely the union of all the simple closed curves bounding the disks used in the construction. Except for a finite number of points of the boundary of an admissible region, the boundary in a sufficiently small neighborhood of a boundary point is an arc. The finite number of exceptional points will be called *junction* points.

THEOREM 4. *If R is an admissible region in the sense defined above, then $m(R) < \mu(R) < M(R)$.*

In view of Theorem 3 it suffices to prove $m(R) < \mu(R)$. We begin the proof with two lemmas.

LEMMA 5. *If S is any finite set of points in the plane, and P_1 is a point not in S, then a coordinate system can be imposed so that P_1 is a lattice point but no point of S is a lattice point.*

PROOF. Let a coordinate system be imposed so that P_1 is a lattice point. The set of points S can be separated into two disjoint sets S_1 and S_2, where the members of S_1 are lattice points but the members of S_2 are not. Let δ be a positive real number smaller than the shortest distance of any point of S_2 to a lattice point. Rotate the coordinate system, with P_1 as center of rotation, through a sufficiently small angle so that no lattice point in the vicinity of any point of S_2 is moved by more than a distance δ, and so that the points of S_1 are no longer lattice points. Such a rotation gives a coordinate system satisfying the conditions of the lemma.

LEMMA 6. *If R is an admissible region, a coordinate system can be imposed so that one and only one lattice point lies on the boundary of R, and it is not a junction point of the boundary of R.*

PROOF. Let S be the set of junction points of R, and let P_1 be any point on the boundary of R, with P_1 not in S. Then by Lemma 5 a coordinate system C_1 can be imposed so that P_1 is a lattice point, and so that no point of S is a lattice point. Suppose that all the lattice points on the boundary of R are $P_1, P_2, \cdots, P_r$ with $r \geqq 2$. None of these points is a junction point. Choose $\delta > 0$ so that except for the points $P_1, P_2, \cdots, P_r$ the distance from every lattice point to the boundary of R exceeds δ. We also choose δ to be less than the distance from any of $P_1, P_2, \cdots, P_r$ to any junction point on the boundary of R. We separate the proof into two cases, $r=2$ and $r>2$.

If $r=2$ we try to remove P_2 from the boundary by a small rotation of the coordinate system with center of rotation P_1. This might not succeed because the boundary of R in the vicinity of P_2 might be a circular arc with center P_1. If so, then starting from C_1 again we try to remove P_1 from the boundary of R by a small rotation of the coordinate system with center of rotation P_2. If this does not remove P_1 from the boundary then we know that the boundary of R in the neighborhoods of P_1 and P_2 consists of two circular arcs with centers at P_2 and P_1 respectively.

In this case we move the entire coordinate system from its original position C_1 along a path parallel to the circular arc at P_1, starting at P_1 and moving along the circular arc in one direction. In this motion of the coordinate system, no rotation is involved. This must remove the lattice point at P_2 from the boundary of R, since otherwise this would mean that the boundary of R in the neighborhood of P_2 would consist of a small circular arc intersected by another. But this contradicts the fact that the local boundary of R is a simple closed curve in the neighborhood of P_2. These motions of the coordinate system can be sufficiently small so that the total distance moved by any lattice point in the vicinity of R is less than δ, and so we get exactly one lattice point, P_1 or P_2, on the boundary. And this lattice point is not a junction point.

Next we turn to the other case of the proof, $r>2$. Rotate the coordinate system, with P_1 as center of rotation, so that no lattice point in the vicinity of R moves a greater distance than $\delta/2$. If in the process of rotation the lattice points $P_2, P_3, \cdots, P_r$ move off the boundary, the proof of the lemma is complete in this case. If not, we may presume that for a rotation about P_1 through some small angle the points $P_2, \cdots, P_k$, with $k \leqq r$, remain on the boundary of R. (As to notation here, the labels P_2 etc. stay with the lattice points as they move.) From the original position of the coordinate system, say C_1, we may presume that we have moved to a position C_2 with lattice points $P_1, P_2, \cdots, P_k$ on the boundary, and such that a small rotation of the coordinate system *in both directions* with P_1 as center of rotation does not move $P_2, \cdots, P_k$ off the boundary of R.

Next, starting from the coordinate system position C_2, make a small rotation with P_2 as center. This rotation will move $P_3, \cdots, P_k$ off the boundary for the following reason. If P_3 for example stayed on the boundary of the region R, this would mean that the local boundary in the neighborhood of P_3 would consist of a small circular arc with center at P_1 (relative to position of axes C_1) intersected by a small circular arc with center P_2 (relative to position of axes C_2). But the local boundary of R at P_3 is a simple closed curve, so this is impossible. Thus there remain only two lattice points, P_1 and P_2, on the boundary of R. But this case has already been treated, so the proof of Lemma 6 is complete.

We now use Lemma 6 to prove Theorem 4. We impose a coordinate system so that exactly one lattice point, say P, lies on the boundary of R, and P is not a junction point. Now the boundary of R in the neighborhood of P is an arc (of a disk) and so there are points P_1 interior to R and P_2 exterior to R such that the points of the straight line segment PP_1 belong to R, but the points of the straight line segment PP_2 (except P itself) are not in R. By translating the coordinate system twice, once in the direction PP_1 and once in the direction PP_2, we can move this lattice point P off the boundary, relocating it either inside R or outside R. This can be done without disturbing the in-or-out relationship of all the lattice points in the plane other than P.

Denote the original coordinate system, with P a lattice point on the boundary of R, by Γ. Denote the coordinate system resulting from a small translation of Γ in the direction PP_1 by Γ_1, and similarly the system resulting from a small translation of Γ in the direction PP_2 by Γ_2. Let (x_2, y_2) be the coordinates of the origin of the Γ_2 system when related to the Γ_1 system. Let the function $C(R, x, y)$ as defined in the proof of Theorem 2 refer to the Γ_1 coordinate system. Then we have

$$C(R, 0, 0) = C(R, x_2, y_2) + 1 \geqq m(R) + 1.$$

Furthermore with the Γ_1 coordinate system there are no lattice points on the boundary of R, and there are $C(R, 0, 0)$ lattice points in the interior of R. Let $\delta > 0$ be chosen so that for every one of these

$C(R, 0, 0)$ lattice points, a disk of radius δ with center at the lattice point lies entirely inside R. Then for each (x, y) satisfying $x^2+y^2 \leqq \delta^2$ we see that $C(R, x, y) = C(R, 0, 0)$. With T_1 and T_2 defined as in (4) and (5) we use (3) to get

$$\begin{aligned}
\mu(R) &= \iint_{T_1} C(R, x, y)dxdy + \iint_{T_2} C(R, x, y)dxdy \\
&= \iint_{T_1} C(R, 0, 0)dxdy + \iint_{T_2} C(R, x, y)dxdy \\
&\geqq \iint_{T_1} \{m(R) + 1\}dxdy + \iint_{T_2} m(R)dxdy \\
&= m(R) + \iint_{T_1} dxdy > m(R),
\end{aligned}$$

as asserted in Theorem 4.

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