BOUNDS FOR SOLUTIONS OF 2ND ORDER COMPLEX DIFFERENTIAL EQUATIONS

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1. It is well known [1], [2] that upper and lower bounds for the norm of a solution of ordinary differential systems can be given in terms of solutions of related first order scalar equations. However, the independent variable t is taken to be real there. In [3] upper bounds for solutions of a class of 2nd order complex differential equations were obtained.

In this paper we derive upper as well as lower bounds for solutions of the complex differential equation

$$y'' + y + yf(y, y', z) = 0, \tag{1}$$

where f is an entire function of y and y', analytic in z for $|z| < R$.

Let Y denote the column vector with components y, y' and let $\bar{f}$ denote the function of Y and z which takes the values $f(y, y', z)$, that is,

$$\bar{f}(Y, z) = f(y, y', z).$$

(1) is equivalent to

$$Y' = AY + B(Y, z)Y, \tag{2}$$

where A, $B(Y, z)$ are the matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ -\bar{f} & 0 \end{pmatrix}$$

respectively.

2. We use the absolute value norm; namely, for a vector Y with components y, y',

$$|Y| = |y| + |y'|.$$

LEMMA 1. *Suppose that there is a continuous, nonnegative function $g(s, t)$ defined on the half-strip $\{(s, t) | 0 \leqq s < \infty, 0 \leqq t < R\}$, such that*

$$|\bar{f}(Y, z)| \leqq g(|Y|, |z|). \tag{3}$$

Let $y(z)$ be a solution of (1) *for which*

$$|y(0)| = a, \quad |y'(0)| = b, \quad a + b > 0,$$

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and let $s(t)$ *be the maximal solution of*

$$ds/dt = s(1 + g(s, t)), \tag{4}$$

satisfying $s(0) = a+b$. *Then, for all* t $(<R)$ *such that* $s(t)$ *exists,* R *being assumed sufficiently large, we have*

$$|Y(z)| \leqq s(t), \qquad t = |z|.$$

PROOF. Let $\Phi(z)$ be the fundamental matrix of $Y' = AY$; that is,

$$\Phi(z) = \begin{pmatrix} \cos z & \sin z \\ -\sin z & \cos z \end{pmatrix}.$$

A solution of (2) which satisfies $Y(0) = Y_0$ is

$$Y(z) = \Phi(z) Y_0 + \int_0^z \Phi(z)\Phi^{-1}(\xi) B(Y(\xi), \xi) Y(\xi) d\xi, \tag{5}$$

where the integration is carried out along the ray $\theta = \theta_0$. Let us write

$$z = t\exp(i\theta_0), \qquad \xi = \tau \exp(i\theta_0).$$

Then (5) can be written as

$$Y(t, \theta_0) = \Phi(t, \theta_0) Y_0 + \exp(i\theta_0) \cdot \int_0^t \Phi(t, \theta_0)\Phi^{-1}(\tau, \theta_0) B(Y(\tau, \theta_0), \tau\exp(i\theta_0)) Y(\tau, \theta_0) d\tau,$$

where $Y(\cdot, \exp(i\theta_0)) \equiv Y(\cdot, \theta_0)$. Also, if $h > 0$,

$$Y(t+h, \theta_0) = \Phi(t+h, \theta_0) Y_0 + \exp(i\theta_0) \cdot \int_0^{t+h} \Phi(t+h, \theta_0)\Phi^{-1}(\tau, \theta_0) B(Y(\tau, \theta_0), \tau\exp(i\theta_0)) Y(\tau, \theta_0) d\tau.$$

If we let $m(t, \theta_0) = |Y(t, \theta_0)|$, then

$$\begin{aligned} m(t+h, \theta_0) - m(t, \theta_0) &\leqq |Y(t+h, \theta_0) - Y(t, \theta_0)| \\ &\leqq |Y(t+h, \theta_0) - \Phi(t+h, \theta_0)\Phi^{-1}(t, \theta_0) Y(t, \theta_0)| \\ &\quad + |\Phi(t+h, \theta_0)(\Phi^{-1}(t+h, \theta_0) - \Phi^{-1}(t, \theta_0)) Y(t, \theta_0)| \\ &= \left| \Phi(t+h, \theta_0) \int_t^{t+h} \Phi^{-1}(\tau, \theta_0) B(Y(\tau, \theta_0), \tau\exp(i\theta_0)) Y(\tau, \theta_0) d\tau \right| \\ &\quad + |\Phi(t+h, \theta_0)(\Phi^{-1}(t+h, \theta_0) - \Phi^{-1}(t, \theta_0)) Y(t, \theta_0)|. \end{aligned}$$

Since

$$\Phi(t, \theta_0) \frac{d\Phi^{-1}}{dt} (t, \theta_0) Y(t, \theta_0)$$

$$= - \exp(i\theta_0) A y(t, \theta_0) = \exp(i\theta_0) \begin{pmatrix} -y'(t \exp(i\theta_0)) \\ y(t \exp(i\theta_0)) \end{pmatrix},$$

we get

$$\dot{m}_+(t, \theta_0) \leqq m(t, \theta_0)(1 + g(m(t, \theta_0), t)), \tag{6}$$

where $\dot{m}_+$ is the right-hand derivative of m.

Hence the conclusion follows from Theorem 4.1, p. 26 of [4], in view of the arbitrariness of θ_0.

THEOREM 2. *Let the hypotheses of Lemma* 1 *be satisfied. Then,*

$$|y(z)| \leqq e^{-t}\left[a + \int_0^t s(\tau) e^{\tau} d\tau\right], \qquad t = |z|.$$

PROOF. By Lemma 1,

$$|y(\tau \exp(i\theta_0))| + |y'(\tau \exp(i\theta_0))| = |Y(\tau, \theta_0)| \leqq s(\tau),$$

and so

$$|y(\tau \exp(i\theta_0))| + \frac{d}{d\tau}(|y(\tau \exp(i\theta_0))|) \leqq s(\tau).$$

Therefore,

$$\frac{d}{d\tau}(e^{\tau} |y(\tau \exp(i\theta_0))|) \leqq e^{\tau} s(\tau),$$

whence,

$$e^t |y(t \exp(i\theta_0))| \leqq a + \int_0^t s(\tau) e^{\tau} d\tau.$$

The conclusion follows since θ_0 is arbitrary.

EXAMPLE. If $g(s, t)$ is of the form ks^n, (4) becomes

$$ds/dt = s + ks^{n+1},$$

which can be solved explicitly. Indeed, the solution satisfying $s(0) = a + b$ is

$$e^t[((a + b)^{-n} + k) - ke^{nt}]^{-1/n}.$$

Thus, in particular, when $k = n = a + b = 1$, we have for $t < \ln 2$,

$$|y(z)| \leqq (a+1)e^{-t} - 1 - 2e^{-t}\ln(2-e^{t}).$$

3. In addition to the estimate (6), we get

$$-m(t, \theta_0)(1 + g(m(t, \theta_0), t)) \leqq \dot{m}_+(t, \theta_0). \tag{7}$$

This leads to

LEMMA 3. *Let $y(z)$ be a solution of* (1) *as in Lemma* 1. *Let $\sigma(t)$ be the minimal solution of*

$$d\sigma/dt = -\sigma(1 + g(\sigma, t)), \tag{8}$$

satisfying $\sigma(0)=a+b$. Then, for all t $(<R)$ such that $\sigma(t)\geqq 0$, we have

$$|Y(z)| \geqq \sigma(t), \qquad t = |z|. \tag{9}$$

PROOF. It is sufficient to show that, for arbitrary θ_0,

$$m(t, \theta_0) \geqq \sigma_\epsilon(t), \tag{10}$$

where $\sigma_\epsilon(t)$ is a solution of

$$d\sigma/dt = -\sigma(1 + g(\sigma, t)) - \epsilon, \qquad \epsilon > 0,$$

satisfying the same initial condition as $\sigma(t)$.

Suppose for some $\epsilon>0$, (10) is false. Then there exists $\hat{t}$ $(\geqq 0)$ such that

$$m(\hat{t}, \theta_0) = \sigma_\epsilon(\tilde{t}), \qquad m(t, \theta_0) < \sigma_\epsilon(t) \quad \text{for } t > \hat{t};$$

whence,

$$\dot{m}_+(t, \theta_0) \leqq (d\sigma_\epsilon/dt)(\hat{t}) = -m(\hat{t}, \theta_0)(1 - g(m(\hat{t}, \theta_0), \hat{t})) - \epsilon,$$

a contradiction in view of (7). This completes the proof.

Before we turn to the main theorem of this section, we state, as a separate lemma, the following result which we require.

LEMMA 4. *Let $|y(z)| = \rho(t, \theta)$, $z = te^{i\theta}$, and $M(t) = \max_{0\leq\theta\leq 2\pi} \rho(t, \theta)$. Then,*

$$(\partial\rho/\partial t)(t, \theta_0) \leqq \dot{M}_+(t),$$

where $M(t) = \rho(t, \theta_0)$.

PROOF. Let $h>0$ and let $M(t+h) = \rho(t+h, \theta_h)$. Then,

$$\frac{\rho(t+h, \theta_0) - \rho(t, \theta_0)}{h} \leqq \frac{M(t+h) - M(t)}{h}.$$

The conclusion is immediate in view of the fact that $\partial\rho/\partial t$ exists.

THEOREM 5. *Let the hypotheses of Lemma 3 be satisfied. Then,*

$$M(t) \geqq e^{-t}\left[a + \int_0^t \sigma(\tau)e^{\tau}d\tau\right], \qquad t = |z|. \tag{11}$$

PROOF. Set

$$y(z) = \rho e^{i\phi}, \qquad z = te^{i\theta}.$$

For each t,

$$ite^{i\theta}y'(z) = \frac{\partial}{\partial\theta}(\rho e^{i\phi}) = e^{i\phi}\left[\frac{\partial\rho}{\partial\theta} + i\rho\frac{\partial\phi}{\partial\theta}\right]. \tag{12}$$

If, for fixed t, the maximum $M(t)$ of $\rho(t, \theta)$ is taken when $\theta=\theta_0$, we have

$$(\partial\rho/\partial\theta)(t, \theta_0) = 0.$$

Therefore, (12) yields

$$t\exp(i\theta_0)y'(z_0) = \rho e^{i\phi}(\partial\phi/\partial\theta)(t, \theta_0), \qquad z_0 = t\exp(i\theta_0),$$

that is, $\exp(i(\theta_0-\phi))y'(z_0)=(\partial\rho/\partial t)(t, \theta_0)$, by the Cauchy-Riemann equations. Also, since $\partial\rho/\partial t$ exists, it is easy to see that

$$(\partial\rho/\partial t)(t, \theta_0) \geqq 0;$$

and so,

$$|y'(z_0)| = (\partial\rho/\partial t)(t, \theta_0).$$

Thus, from (9) and Lemma 4,

$$M(t) + \dot{M}_+(t) \geqq \sigma(t),$$

that is,

$$\dot{K}_+(t) \geqq e^t\sigma(t), \qquad K(t) = e^tM(t). \tag{13}$$

The proof is completed by noting that (11) is the integral form of (13) since $K(0)=a$ and $t\geqq 0$.

EXAMPLE. As earlier, if $g(s, t)$ is ks^{n+1}, (8) is

$$d\sigma/dt = -\sigma - k\sigma^{n+1}.$$

The solution of this equation for which $\sigma(0)=a+b$ is

$$e^{-t}[((a+b)^{-n}+k) - ke^{-nt}]^{-1/n}.$$

Thus, when $k=n=a+b=1$, we have

$$M(t) \geqq e^{-t}\left[a + \tfrac{1}{2}\ln(2e^{t} - 1)\right].$$

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