STOKES MULTIPLIERS OF SUBDOMINANT SOLUTIONS OF THE DIFFERENTIAL EQUATION $y''-(x^3+\lambda)y=0$

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1. **Introduction.** The Stokes multipliers of subdominant solutions of the differential equation $y''-(x^m+\lambda)y=0$, where $''=d^2/dx^2$ and λ is a parameter, are well known in case $m=1$ or $m=2$. The subdominant solutions of this differential equation are essentially Airy functions for $m=1$ and parabolic cylinder functions for $m=2$ and these functions have been studied extensively [**1**, p. 446 and p. 686]. However, for integral m greater than two, there seems to be no special functions with which subdominant solutions can be identified. Further, the techniques for obtaining the Stokes multipliers in the cases $m=1$, 2 are based on detailed knowledge of the structure of the subdominant solutions, for example integral representations.

In this note we shall treat the differential equation, $m=3$,

$$y'' - (x^3 + \lambda)y = 0,$$

by a method which is not based on the detailed structure of any particular special functions. From the discussion it will be apparent that the method can be applied to more general cases.

For a discussion of the methods that have been utilized in studying Stokes multipliers as numerical constants see H. L. Turrittin [**4**]. However, the study of the dependence of Stokes multipliers on parameters seems to be rather new.

Let us consider the differential equation

$$y'' - (x^3 + \lambda)y = 0, \tag{1.1}$$

where x is the complex independent variable, λ is a complex parameter, and y is an unknown function of x and λ. By application of results of P. F. Hsieh and Y. Sibuya [**2**] for differential equations of the form $y''-P(x)y=0$, $P(x)$ a polynomial of x, we can derive the following theorem:

THEOREM 1. *Equation* (1.1) *has a solution*

$$y = f(x, \lambda) \tag{1.2}$$

such that (i) f *is an entire function of* x *and* λ; (ii) f *and* f' *admit respec-*

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tively the asymptotic representations

$$f(x, \lambda) = x^{-3/4}\{1 + O(x^{-1/2})\} \exp\{-(2/5)x^{5/2}\},$$
$$f'(x, \lambda) = x^{3/4}\{-1 + O(x^{-1/2})\} \exp\{-(2/5)x^{5/2}\}, \tag{1.3}$$

uniformly on each compact set in the λ*-plane as* x *tends to infinity in any sector of the form:*

$$|\arg x| \leqq 3\pi/5 - \rho_0, \tag{1.4}$$

where ρ_0 *is an arbitrary positive number, and*

$$\arg(x^r) = r \arg x \tag{1.5}$$

for any constant r.

Let us denote by S_j ($j=0, \pm 1, \pm 2$) the sectors defined respectively by

$$|\arg x - 2j\pi/5| < \pi/5. \tag{1.6-j}$$

Put

$$\omega = \exp(2\pi i/5). \tag{1.7}$$

Then the function $f(\omega^{-j}x, \omega^{2j}\lambda)$ satisfies equation (1.1) and this solution tends to zero as x tends to infinity along any direction in the sector S_j. Hence the solution

$$y = f(\omega^{-j}x, \omega^{2j}\lambda) \tag{1.8-j}$$

is said to be subdominant in the sector S_j. In particular the solution (1.2) is subdominant in the sector S_0. It is easily seen that the two solutions (1.8-1) and (1.8-2) are linearly independent. Hence the solution (1.2) is a linear combination of those two solutions. Put

$$f(x, \lambda) = c_1(\lambda)f(\omega^{-1}x, \omega^2\lambda) + c_2(\lambda)f(\omega^{-2}x, \omega^4\lambda). \tag{1.9}$$

In order to study the asymptotic behavior of $f(x, \lambda)$ as x tends to infinity in the sector S_2, we must compute c_1 and c_2. These quantities are Stokes multipliers of the solution $f(x, \lambda)$. First of all we shall show that $c_2(\lambda)$ is identically equal to $-\omega$. Then we shall derive asymptotic representations of $c_1(\lambda)$ as λ tends to infinity in any given direction.

2. **Derivation of** $c_2(\lambda)$. In this section we shall show that $c_2(\lambda)$ is identically equal to $-\omega$. In fact, the solution (1.8-1) and its derivative admit respectively the asymptotic representations

$$f(\omega^{-1}x, \omega^2\lambda) = \omega^{3/4}x^{-3/4}\{1 + O(x^{-1/2})\} \exp\{(2/5)x^{5/2}\},$$
$$\omega^{-1}f'(\omega^{-1}x, \omega^2\lambda) = \omega^{3/4}x^{3/4}\{1 + O(x^{-1/2})\} \exp\{(2/5)x^{5/2}\} \tag{2.1}$$

uniformly on each compact set in the λ-plane as x tends to infinity in any sector of the form $|\arg x-2\pi/5|\leqq 3\pi/5-\rho_0$. Since this sector and the sector (1.4) have a common part, and the Wronskian of any two solutions of equation (1.1) is independent of x, we can compute the Wronskian Δ of $f(x, \lambda)$ and $f(\omega^{-1}x, \omega^2\lambda)$ from (1.3) and (2.1) by letting x tend to infinity; namely we have

$$\Delta = \begin{vmatrix} f(x, \lambda) & f(\omega^{-1}x, \omega^2\lambda) \\ f'(x, \lambda) & \omega^{-1}f'(\omega^{-1}x, \omega^2\lambda) \end{vmatrix} = 2\omega^{3/4}. \tag{2.2}$$

From (1.9) it is easy to derive $\Delta = -\omega^{-1}c_2(\lambda)\Delta$, hence we get $c_2(\lambda) = -\omega$.

3. **Asymptotic representation of $c_1(\lambda)$.** Let us denote by $D(\lambda)$ the Wronskian of $f(x, \lambda)$ and $f(\omega^{-2}x, \omega^4\lambda)$. Then from (2.2) and

$$f(x, \lambda) = c_1(\lambda)f(\omega^{-1}x, \omega^2\lambda) - \omega f(\omega^{-2}x, \omega^4\lambda) \tag{3.1}$$

we can derive

$$D(\lambda) = 2\omega^{-1/4}c_1(\lambda). \tag{3.2}$$

Hence $c_1(\lambda)$ is an entire function of λ. We shall study $D(\lambda)$ instead of $c_1(\lambda)$ as λ tends to infinity. The Wronskian $D(\lambda)$ being independent of x, we get

$$D(\lambda) = \omega^{-2}f'(0, \omega^4\lambda)f(0, \lambda) - f(0, \omega^4\lambda)f'(0, \lambda). \tag{3.3}$$

This suggests that we start with an asymptotic evaluation of $f(0, \lambda)$ and $f'(0, \lambda)$. Such a result is given in the following theorem:

THEOREM 2. *Let δ_0 be an arbitrary positive number. Then as λ tends to infinity in the sector*

$$|\arg \lambda| \leqq \pi - \delta_0, \tag{3.4}$$

the quantities $f(0, \lambda)$ and $f'(0, \lambda)$ admit respectively the asymptotic representations

$$\begin{aligned} f(0, \lambda) &= \lambda^{-1/4}\{1 + o(1)\} \exp\{K\lambda^{5/6}\}, \\ f'(0, \lambda) &= \lambda^{1/4}\{-1 + o(1)\} \exp\{K\lambda^{5/6}\}, \end{aligned} \tag{3.5}$$

where $\arg(\lambda^\gamma) = \gamma \arg \lambda$ for any constant γ and

$$K = \int_0^{+\infty} \{(t^3+1)^{1/2} - t^{3/2}\}\, dt.$$

This theorem can be proved by reconstructing the solution $f(x, \lambda)$ along the positive real axis in the x-plane. A method for such a computation is available in the book of M. A. Neumark [**3**, pp. 257–260 and pp. 279–288]. Hence we shall omit the proof of Theorem 2.

Now if we assume that λ is in the sector

$$-3\pi/5 + \delta_0 \leqq \arg \lambda \leqq \pi - \delta_0, \tag{3.6}$$

the quantities λ and $\omega^{-1}\lambda$ satisfy the condition (3.4). Hence

$$\begin{aligned} f(0, \omega^4\lambda) &= \omega^{1/4}\lambda^{-1/4}\{1 + o(1)\} \exp \{K\omega^{-5/6}\lambda^{5/6}\}, \\ f'(0, \omega^4\lambda) &= \omega^{-1/4}\lambda^{1/4}\{-1 + o(1)\} \exp \{K\omega^{-5/6}\lambda^{5/6}\} \end{aligned} \tag{3.7}$$

as λ tends to infinity. Inserting (3.5) and (3.7) into (3.3), we can prove the following theorem:

THEOREM 3. *As λ tends to infinity in the sector* (3.6), *the quantity $c_1(\lambda)$ admits the asymptotic representation*

$$c_1(\lambda) = \omega^{-2}\{-1 + o(1)\} \exp \{K(1 + \omega^{-5/6})\lambda^{5/6}\}. \tag{3.8}$$

This straightforward method can not be used in any sector other than the sector (3.6). For example, in the sector

$$-\pi + \delta_0 \leqq \arg \lambda \leqq -3\pi/5 - \delta_0,$$

the quantities λ and $\omega^4\lambda$ satisfy the condition (3.4). Hence

$$\begin{aligned} f(0, \omega^4\lambda) &= \omega^{-1}\lambda^{-1/4}\{1 + o(1)\} \exp \{K\omega^{10/3}\lambda^{5/6}\}, \\ f'(0, \omega^4\lambda) &= \omega\lambda^{1/4}\{-1 + o(1)\} \exp \{K\omega^{10/3}\lambda^{5/6}\}. \end{aligned}$$

If we try to use these expressions for the computation of $D(\lambda)$, we obtain

$$D(\lambda) = \{\omega^{-2}\omega - \omega^{-1} + o(1)\} \exp \{K(1 + \omega^{10/3})\lambda^{5/6}\}.$$

But since $\omega^{-2}\omega - \omega^{-1} = 0$, these expressions are not suitable for the computation of $D(\lambda)$. Therefore we need more precise representations of $f(0, \lambda)$ and $f'(0, \lambda)$. To derive such representations, we use Theorem 3. First of all, we replace λ by $\omega^3\lambda$ in formula (3.1). Then we get

$$\begin{aligned} f(0, \lambda) &= \{c_1(\omega^3\lambda)\}^{-1}\{f(0, \omega^3\lambda) + \omega f(0, \omega^2\lambda)\}, \\ f'(0, \lambda) &= \{c_1(\omega^3\lambda)\}^{-1}\{\omega f'(0, \omega^3\lambda) + f'(0, \omega^2\lambda)\}. \end{aligned} \tag{3.9}$$

Notice that, if λ is in the sector

$$-\pi - 4\pi/5 + \delta_0 \leqq \arg \lambda \leqq -3\pi/5 - \delta_0, \tag{3.10}$$

then the quantities $\omega^3\lambda$ and $\omega^4\lambda$ satisfy the condition (3.4), and furthermore the quantity $\omega^3\lambda$ satisfies the condition (3.6). Hence from formula (2.2), Theorems 2 and 3, we can derive the following theorem:

THEOREM 4. *As* λ *tends to infinity in the sector* (3.10), *the quantity* $c_1(\lambda)$ *admits the asymptotic representation*

$$c_1(\lambda) = \omega^{-2}[\{-1 + o(1)\} \exp \{K(\omega^{-5/6} - \omega^{5/6})\lambda^{5/6}\} + \{-1 + o(1)\} \exp \{K(1 + \omega^{-5/6})\lambda^{5/6}\}]. \tag{3.11}$$

In the proof of Theorem 4, we use the identities $\omega^{1/2} = -\omega^{-2}$ and $\omega^3 = \omega^{-2}$. The first term of (3.11) dominates the second term if $\arg \lambda < -4\pi/5$, and the second term dominates the first term if $\arg \lambda > -4\pi/5$. Along the direction $\arg \lambda = -4\pi/5$ we can locate zeros of $c_1(\lambda)$ by using (3.11). If λ_0 is one of those zeros of $c_1(\lambda)$, then the solution $f(x, \lambda_0)$ is subdominant in the sector S_2 as well as in S_0.

Finally we consider the sector

$$-\pi + \delta_0 \leqq \arg \lambda \leqq \pi/5 - \delta_0. \tag{3.12}$$

In this case, we replace λ by $\omega^4\lambda$ in formula (3.9). Then

$$\begin{aligned} f(0, \omega^4\lambda) &= \{c_1(\omega^2\lambda)\}^{-1}\{f(0, \omega^2\lambda) + \omega f(0, \omega\lambda)\}, \\ f'(0, \omega^4\lambda) &= \{c_1(\omega^2\lambda)\}^{-1}\{\omega f'(0, \omega^2\lambda) + f'(0, \omega\lambda)\}. \end{aligned} \tag{3.13}$$

Notice that, if λ is in the sector (3.12), the quantities λ and $\omega\lambda$ satisfy the condition (3.4), and furthermore the quantity $\omega^2\lambda$ satisfies the condition (3.6). Hence from formula (2.2), Theorems 2 and 3, we can derive the following theorem:

THEOREM 5. *As* λ *tends to infinity in the sector* (3.12), *the quantity* $c_1(\lambda)$ *admits the asymptotic representation* (3.11).

The three sectors (3.6), (3.10) and (3.12) cover the λ-plane completely. Thus the behavior of $c_1(\lambda)$ as λ tends to infinity in any given direction can be given by one of Theorems 3, 4 and 5.

REMARK. The asymptotic representations of $f(0, \lambda)$ and $f'(0, \lambda)$ in the sector

$$|\arg \lambda + \pi| \leqq \delta_0 \tag{3.14}$$

can also be derived from (3.9). We merely state the results:

The quantities $f(0, \lambda)$ *and* $f'(0, \lambda)$ *admit respectively the asymptotic representations*

$$f(0, \lambda) = -i\lambda^{-1/4}\{1 + o(1)\}[\exp\{-K\lambda^{5/6}\} + i \exp\{K\omega^{5/3}\lambda^{5/6}\}] \cdot \exp\{K(1 + \omega^{-5/6})\lambda^{5/6}\},$$

$$f'(0, \lambda) = i\lambda^{1/4}\{1 + o(1)\}[-\exp\{-K\lambda^{5/6}\} + i \exp\{K\omega^{5/3}\lambda^{5/6}\}] \cdot \exp\{K(1 + \omega^{-5/6})\lambda^{5/6}\}$$

as λ *tends to infinity in the sector* (3.14).

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CORRECTION TO "PERIODIC SOLUTIONS OF FOURTH-ORDER DIFFERENTIAL EQUATIONS"

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In the proof of the theorem in [1] we need the fact that the solutions of

$$x' = f(x, \mathfrak{y}(t)) \tag{3}$$

are defined for all $t \geqq 0$. The argument that is given, namely that we can assume that $\|f(x, y)\| \leqq 1$, is not correct. Therefore an extra hypothesis is needed to insure the existence of solutions of (3) for all $t \geqq 0$. For example, this global existence property would be satisfied if one assumed that for every compact set $K \subset R^2$ there exist constants M and B such that

$$\|f(x, y)\| \leqq M\|x\| + B \qquad (x \in R^2, y \in K).$$

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Reference

1. G. R. Sell, *Periodic solutions of fourth-order differential equations*, Proc. Amer. Math. Soc. **17** (1966), 808–809.

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