A DENSITY THEOREM FOR WALSH FUNCTIONS

J. J. PRICE

1. **Introduction.** A theorem of Levinson [3, p. 13] on the closure of sets of exponentials contains the following result. Let $S=\{n_j\}$ be an increasing sequence of positive integers. Define

$$D(S) = \limsup_{\xi\to 1-0} \limsup_{n\to\infty} (\Lambda(n) - \Lambda(\xi n))/(n - \xi n)$$

where $\Lambda(n)$ is the number of elements of S less than n. Suppose $D(S)=1$. Then for each positive ϵ, there is a subset E_ϵ of the unit interval whose measure exceeds $1-\epsilon$ and such that the family $[\exp(2\pi i n_j x)\colon n \in S]$ is total in $L^2(E_\epsilon)$.

The proof of Levinson's theorem requires deep complex methods. In this paper, we obtain an analogous result for Walsh functions using elementary methods. Our result requires the weaker hypothesis that $\rho(S)=1$ where

$$\rho(S) = \limsup_{k\to\infty} \limsup_{n\to\infty} (\Lambda(n + k) - \Lambda(n))/k.$$

It is an exercise to verify that for every sequence S, $\rho(S)\geqq D(S)$. Equality need not hold. For example, if S is the sequence of all integers n such that $2^k\leqq n<2^k+k$ for some k, then $\rho(S)=1$ but $D(S)=0$.

For definitions and properties of Walsh functions, see N. J. Fine [1].

2. **Preliminaries.** First some notation. All numbers mentioned will be nonnegative integers. The unit interval $[0, 1)$ will be denoted by I, the dyadic interval $[q2^{-p}, (q+1)2^{-p})$ by $I(p, q)$. Associated with $I(p, q)$ are its characteristic functions, denoted by $\psi(p, q)$ and the Haar function supported on $I(p, q)$, denoted by $h(p, q)$. (For definition of Haar functions see [2].)

In the next section we use two results which we quote here. Both are special cases of known theorems, the first by A. A. Talalyan [5] and the second by Robert E. Zink and the author [4].

THEOREM 1. *Let Φ be a sequence of real functions defined a.e. on I. Then Φ is total in measure on I if and only if for each positive ϵ there is a subset E_ϵ of I whose measure exceeds $1-\epsilon$ such that Φ is total in $L^2(E_\epsilon)$.*

Received by the editors February 3, 1966.

THEOREM 2. *Let $\Phi=\{\phi_n\}$ be a sequence of Haar functions. For each n let E_n be the support of ϕ_n. Let $E=\limsup E_n$. Then Φ is total in measure on I if and only if $I-E$ is a null set.*

3. A density theorem for Walsh functions.

THEOREM. *Let $S=\{n_j\}$ be an increasing sequence of positive integers and let Ψ_S be the family of Walsh functions $[\psi_{n_j}: n_j\in S]$. If $\rho(S)=1$, then for every positive ϵ there is a subset E_ϵ of I whose measure exceeds $1-\epsilon$ such that Ψ_S is total in $L^2(E_\epsilon)$.*

PROOF. By virtue of Theorem 1, it suffices to prove Ψ_S is total in measure on I. This would follow from the existence of a measure preserving transformation T of I onto itself that is one-to-one except for a denumerable set of points and such that the family $\Psi'_S = [\psi_{n_j}\circ T: n_j\in S]$ is total in measure on I. By virtue of Theorem 2, it suffices to show the existence of a transformation T such that the linear span of Ψ'_S contains a family of Haar functions of the form $[h(p_i, q): 0\leq q<2^{p_i}, p_1<p_2<p_3 \cdots]$.

Let p be given. From the assumption $\rho(S)=1$, it follows that there is an increasing sequence $\{j_i\}$ such that S contains all integers n for which $j_i2^p\leq n<(j_i+1)2^p$, or equivalently, such that

$$(1)\qquad (\Lambda((j_i+1)2^p)-\Lambda(j_i2^p))/2^p=1.$$

For if $\rho(S)=1$, it is not hard to verify that also

$$\limsup_{v\to\infty}\ \limsup_{u\to\infty}\ (\Lambda((u+v)2^p)-\Lambda(u2^p))/v2^p=1.$$

Consequently, there exist increasing sequences, $\{u_i\}$ and $\{v_i\}$ with $v_{i+1}>v_i+u_i$ such that

$$(2)\qquad (\Lambda((u_i+v_i)2^p)-\Lambda(u_i2^p))/v_i2^p>1-2^{-p}.$$

At least one of the quantities

$$(3)\qquad (\Lambda((j+1)2^p)-\Lambda(j2^p))/2^p,\qquad v_i\leq j<v_i+u_i,$$

must exceed $1-2^{-p}$; otherwise (2) could not be satisfied. But since the quantities (3) are multiples of 2^{-p}, there exists a value $j=j_i$ for which (1) holds.

Let p_1 be any integer. There is an integer j_1 such that ψ_S contains the set $[\psi_j: j=j_12^{p_1}+m, 0\leq m<2^{p_1}]$. By definition of the Walsh functions, the elements of this set have the form $\psi_{r_1}\psi_m$ where $r_1=j_12^{p_1}$. It is known ([1], p. 373) that the characteristic functions $\chi(p_1, q)$ are linear combinations of the functions $[\psi_m: 0\leq m<2^{p_1}]$. Hence, the linear span of Ψ_S contains the set $[\psi_{r_1}\chi(p_1, q): 0\leq q<2^{p_1}]$.

For some p_2, with $p_2>p_1$, ψ_{r_1} is constant on each interval $I(p_2, s)$.

Each interval $I(p_1, q)$ is partitioned into $2^{p_2-p_1}$ intervals of the form $I(p_2, s)$. On half of these, ψ_{r_1} has the value $+1$; on the others ψ_{r_1} has the value -1. Obviously then, the elements of this partition of $I(p_1, q)$ can be permuted in such a way that $c_1\psi_{r_1}\chi(p_1, q)$ is transformed into the Haar function $h(p_1, q)$, $c_1=2^{(p_1-1)/2}$.

Repeating the argument given above with p_2 in place of p_1, we see that the span of Ψ_S contains $[\psi_{r_2}\chi(p_2, q): 0\leqq q<2^{p_2}]$. The function ψ_{r_2} is constant on intervals $I(p_3, s)$ where $p_3>p_2$.

By induction, we obtain an increasing sequence $\{p_i\}$ such that for each i the span of Ψ_S contains $[\psi_{r_1}\chi(p_i, q): 0\leqq q<2^{p_i}]$.

As noted earlier, there is a transformation T_1 of I onto itself that permutes the intervals $I(p_2, s)$, $0\leqq s<2^{p_2}$, in such a way that the intervals $I(p_1, q)$ are invariant and $c_1\psi_{r_1}\chi(p_1, q)\circ T_1=h(p_1, q)$ for each q, $0\leqq q<2^{p_1}$. There is a permutation T_2 of the intervals $I(p_3, s)$ leaving the intervals $I(p_2, q)$ invariant such that

$$c_i\psi_{r_i}\chi(p_i, q)\circ T_2\circ T_1 = h(p_i, q);$$
$$0\leqq q<2^{p_i},\quad i=1,2,\quad c_i=2^{(pi-1)/2}.$$

By induction, there is a sequence $\{T_n\}$ of transformations of I onto itself such that for each n, T_n permutes the intervals $I(p_{n+1}, s)$ leaving invariant all intervals $I(p_i, q)$, $i\leqq n$, and such that

$$c_i\psi_{r_i}\chi(p_i, q)\circ T_n\circ T_{n-1}\circ\cdots\circ T_1 = h(p_i, q);$$
$$0\leqq q<2^{p_i},\quad i=1,2,\cdots,n.$$

It is a standard fact that the sequence $\{T_n\}$ induces a measure preserving transformation T of I onto itself that is one-to-one except for a countable set of points. Clearly $c_i\psi_{r_i}\chi(p_i, q)\circ T=h(p_i, q)$; $0\leqq q<2^{p_i}$, $i=1, 2, \cdots$. Thus the span of the system $[\psi_{n_j}\circ T: n_j\in S]$ contains a family of Haar functions $[h(p_i, q): 0\leqq q<2^{p_i}, p_1<p_2<\cdots]$. This concludes the proof.

References

1. N. J. Fine, *On the Walsh functions*, Trans. Amer. Math. Soc. **65** (1949), 372–414.

2. S. S. Kaczmarz and H. Steinhaus, *Theorie der Orthogonalreihen*, reprint, Chelsea, New York, 1951.

3. N. Levinson, *Gap and density theorems*, Amer. Math. Soc. Colloq. Publ., Vol. 26, Amer. Math. Soc., Providence, R. I., 1940.

4. J. J. Price and R. E. Zink, *On sets of completeness for families of Haar functions*, Trans. Amer. Math. Soc. 119 (1965), 262–269.

5. A. A. Talalyan, *The representation of measurable functions by series*, Uspehi Mat. Nauk **15** (1960), 77–142 = Russian Math. Surveys **15** (1960), 77–136.

Purdue University