ON CONVEX IDEALS

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In a partially ordered ring, the convex ideals are of special interest because their quotient rings possess a naturally induced order [**3**, 5.2]. Some partially ordered rings have an abundance of convex ideals; for instance, every prime ideal in the ring of all continuous real-valued functions on a topological space is convex (with respect to the usual order) [**3**, 5.5]. The results presented here concern those orders on a ring with respect to which every primitive ideal is convex, or every prime ideal is convex. In particular, we give an example of a commutative partially ordered ring with identity in which every maximal ideal is convex but not every prime ideal is convex.

A partial order on a ring A can be defined by specifying a set $P \subset A$ called the *positive cone*, satisfying the conditions: $P+P \subset P$, $PP \subset P$, and $P \cap -P = \{0\}$. For convenience, we shall identify a partial order on a ring with the positive cone that defines it. An ideal I in A is *P-convex* if $a \in P$, $b-a \in P$, and $b \in I$ imply $a \in I$.

The ring of all continuous real-valued functions on a topological space X is denoted by $C(X)$. We use the symbol R for the field of real numbers, and Q for the field of rational numbers. The one-point compactification of the countable discrete space N is denoted by N*; the point at infinity is designated by ω. The same letter will be used for an ideal I in A and the natural homomorphism of A onto A/I; thus, the image of $a \in A$ in A/I is denoted by $I(a)$.

Observe that $\{0\}$ is a positive cone, and that every ideal in any ring is trivially $\{0\}$-convex. It is evident that the union of a chain of positive cones of a ring is again a positive cone, so that Zorn's Lemma implies the existence of a maximal order on any ring. The following statement is also easily verified.

For any family $\mathfrak{F}$ of ideals in a ring A, let $\mathfrak{C}$ be the collection of orders P on A such that every ideal in $\mathfrak{F}$ is P-convex. Then $\mathfrak{C}$ is nonempty, and every order in $\mathfrak{C}$ is contained in a maximal order in $\mathfrak{C}$.

In particular, there is a maximal order on A with respect to which every primitive ideal in A is convex, and a maximal order on A with respect to which every prime ideal in A is convex.

From now on, we consider mainly commutative semisimple rings.

DEFINITION. Let A be a commutative semisimple ring, and let $\{M_\gamma\}$ be the family of primitive (i.e., prime maximal) ideals in A.

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The *pointwise order* on A corresponding to a family of orders on the quotient fields A/M_γ is the order $\{a \in A: M_\gamma(a) \geqq 0$ in A/M_γ for each $M_\gamma\}$.

It is straightforward to verify that this actually gives an order; it is called the *natural order* in [**2**]. As already indicated, every field can be ordered. The so-called "unorderable" fields are those possessing no total order.

Following Fine, Gillman, and Lambek [**1**, 8.2], we say that an order P on a commutative ring A is *quasi-real* if $a^2 \in P$ for all $a \in A$. Clearly, a pointwise order on a semisimple ring A is quasi-real if and only if it corresponds to a family of quasi-real orders on the quotient fields of A.

Let $\mathfrak{M}$ denote the collection of orders on A with respect to which every primitive ideal in A is convex. Obviously, if $P \in \mathfrak{M}$ and $P' \subset P$, then $P' \in \mathfrak{M}$.

THEOREM. *Let A be a commutative semisimple ring. Every pointwise order on A is in $\mathfrak{M}$; and every order in $\mathfrak{M}$ is contained in a pointwise order. Every maximal order in $\mathfrak{M}$ is a pointwise order that corresponds to a family of maximal orders; and a pointwise order that induces a family of maximal orders is maximal in $\mathfrak{M}$. Every quasi-real maximal order in $\mathfrak{M}$ is a pointwise order that corresponds to a family of total orders; and a pointwise order that induces a family of total orders is quasi-real and maximal in $\mathfrak{M}$.*

PROOF. Let P_0 be any pointwise order on A. For any primitive ideal M in A, if $a \in P_0$ and $b - a \in P_0$, then $0 \leqq M(a) \leqq M(b)$ in the given order on A/M. Thus, $b \in M$ implies $a \in M$, whence M is P_0-convex. Therefore $P_0 \in \mathfrak{M}$.

Now any $P_1 \in \mathfrak{M}$ induces an order on every quotient field of A. Let P_2 be the pointwise order on A corresponding to the resulting family of orders on the quotient fields. For each primitive ideal M in A, $a \in P_1$ implies $M(a) \geqq 0$ in the induced order on A/M; thus, $a \in P_2$. It follows that $P_1 \subset P_2$. (But P_2 need not coincide with P_1; see Remark 5 below.)

It has been shown that every order in $\mathfrak{M}$ is contained in a pointwise order in $\mathfrak{M}$; hence any maximal order in $\mathfrak{M}$ is a pointwise order. Each order in the family to which a maximal order P^* in $\mathfrak{M}$ corresponds can be extended to a maximal order. Then the pointwise order in $\mathfrak{M}$ corresponding to the resulting family contains P^*, and hence coincides with P^*, by maximality. Thus, P^* corresponds to a family of maximal orders.

Next, assume that P is a pointwise order that induces a family of maximal orders, and let P' be a maximal order in $\mathfrak{M}$ containing P. Consider any primitive ideal M in A. Denote the orders on A/M

that P and P' induce by I and I', respectively, and denote the orders on A/M in the families to which P and P' correspond by Q and Q', respectively. If $M(a)\in I$, then there exists $b\in P$ such that $M(b) = M(a)$; and $M(b)\in Q$ by definition of P. It follows that $I\subset Q$. Similarly, $I'\subset Q'$. Also if $M(a)\in I$, then for $b\in P$ such that $M(b)=M(a)$ we have $b\in P'$, whence $M(b)\in I'$. Thus, $I\subset I'$. Since I is maximal by assumption, we have $I=Q$ and $I=Q'$, so that $Q=Q'$. As this holds for all primitive ideals in A, we have $P=P'$. Therefore P is maximal in $\mathfrak{M}$. (In general, I need not coincide with Q; see Remark 6 below.)

The last statement now follows easily from the result that any maximal quasi-real order on a field is total [**1**, 8.5], and the obvious fact that any total order is quasi-real and maximal.

COROLLARY. *The usual order on a ring of continuous functions $C(X)$ is a maximal order with respect to which every maximal ideal is convex.*

PROOF. The orders induced on the quotient fields of a ring of continuous functions $C(X)$ by the usual order are total [**3**; 5.5]; and the usual order is a pointwise order, since a function is nonnegative at each point of X if and only if its image in every quotient field is nonnegative.

REMARKS. (1) It follows immediately from the Corollary and [**3**, 5.5] that the usual order on $C(X)$ is a maximal order with respect to which every prime ideal is convex.

(2) A pointwise order need not be maximal in $\mathfrak{M}$. Any order on a field is a pointwise order; and some fields have orders that are not maximal.

(3) In contrast to [**1**, 8.5], if a maximal order on a field is not quasi-real, it need not be total. Indeed, any field has a maximal order; and some fields have no total order.

(4) Obviously, if A is a commutative semisimple ring such that no quotient field has a nontrivial order, then $\{0\}$ is the only order in $\mathfrak{M}$. The ring of integers is such a ring.

(5) The pointwise order corresponding to a family of orders induced on the quotient fields of A by an order in $\mathfrak{M}$ need not coincide with it. For example, let A be the direct sum of two copies of R, and let $P_1=\{(r,s): r\geqq s\geqq 0\}$ [**3**, 5B.4]. Then the order P_2 corresponding to the family of orders induced on the quotient fields by P_1 is $P_2=\{(r,s): r\geqq 0,\ s\geqq 0\}$. Note also that P_1 is an order in $\mathfrak{M}$ that is not a pointwise order, since the order on a quotient field induced by a pointwise order is contained in the order belonging to the family to which the pointwise order corresponds. We observe that both P_1 and P_2 make A a lattice-ordered ring, providing a counterexample to

[2, 4.1], which was known to be erroneous (Math. Reviews **19** (1958), 1156).

(6) The family of orders induced on the quotient fields by a pointwise order need not coincide with the family of orders to which it corresponds, even if every order in the latter family is total. For example, let θ be the positive square root of 2 (in the usual order on R), and let A be the subring of functions in $C(N^*)$ with values in $Q(\theta)$. We denote the maximal ideal $\{a \in A: a(\omega)=0\}$ by M_ω. Each quotient field of A is isomorphic to $Q(\theta)$. We totally order all quotient fields except A/M_ω by specifying that Q have the usual order and θ be positive; we order A/M_ω similarly but with $-\theta$ positive. If P is the pointwise order on A corresponding to this family of orders, then no element of A whose image in A/M_ω is $-\theta$ can be in P. Thus, $-\theta$ is not positive in the induced order on A/M_ω.

(7) It follows easily from the Theorem that if each quotient field of A has a unique maximal order, then there is a unique maximal order in $\mathfrak{M}$. (An example of a field that has a unique maximal partial order is Q; for further details, see [**5**].) The converse is not true, however. For example, let A be the ring of all functions in $C(N^*)$ that have rational values at the points of N. We denote the maximal ideal $\{a \in A : a(\omega)=0\}$ by J_ω. Then the order P on A induced by the usual order on $C(N^*)$ is the unique maximal order in $\mathfrak{M}$. For, any maximal order in $\mathfrak{M}$ is a pointwise order corresponding to the same family as that to which P corresponds, except perhaps on A/J_ω. But it is easy to see that this implies that the two orders coincide. On the other hand, A/J_ω is isomorphic to R, which does not have a unique maximal partial order; for example, there is a maximal partial order on R in which π is infinitely small relative to Q.

We now generalize the result on prime ideals mentioned in the introduction.

PROPOSITION. *Let I be an ideal in a ring A, where $A = C(X)$ for some topological space X (with the usual order), or A is a commutative regular ring (with a pointwise order). Then every prime ideal in I is convex.*

PROOF. As already mentioned, every prime ideal in $C(X)$ is convex with respect to the usual order. Now suppose A is a commutative regular ring with a pointwise order P. Let J be any ideal in A, and assume that $a \in P$, $b-a \in P$, and $b \in J$. Since A is regular, there exist idempotents e and f in A that generate the same ideals as a and b, respectively. For any primitive ideal M in A, $0 \leqq M(a) \leqq M(b)$ in A/M. Now $M(e)$ and $M(f)$ can only be 0 or 1. It follows that $M(e)=0$ if $M(f)=0$ and $M(f)=1$ if $M(e)=1$. Thus $M(e)=M(e)M(f)$ for each primitive ideal M. Since A is semisimple, we have $e=ef$; this implies that $a\in J$. Therefore J is P-convex.

By [4, 5.1], corresponding to each prime ideal Q in I there is a prime ideal Q^* in A such that $Q^*\cap I=Q$. It is easy to see that the convexity of Q^* implies that of Q.

REMARKS. (8) The condition for convexity of prime ideals given in [1, 9.4] cannot be applied to ideals in $C(X)$; such ideals need not be π-rings (that is, squares need not have positive square roots). For example, the ideal generated by the identity function i in $C(\mathrm{R})$ is not a π-ring, since i^2 has no positive square root in (i). The condition is also inapplicable in the case of a regular ring; an example is easily constructed from a ring having a quotient field with the trivial order.

(9) A prime ideal in a convex ideal need not be convex. For example, let A be the direct sum of the ring of integers with R, and let the order be $\{(n, r): n\geqq 0, r\geqq 0\}$. The ideal I generated by (1, 0) in A is convex; but the prime ideal generated by (2, 0) in I is not convex.

The following example yields more information.

EXAMPLE. *A partially ordered commutative ring with identity such that every maximal ideal is convex but not every prime ideal is convex.* Let $\mathrm{R}[x]$ be the ring of polynomials over R with the usual order, and define A to be the ring of formal power series in y over $\mathrm{R}[x]$ with constant terms in R. Let the order on A be $\{\sum_{i=0}^{\infty} p_i y^i\in A$: the first nonzero p_i is positive$\}$. The unique maximal ideal in A is $\{\sum_{i=0}^{\infty} p_i y^i\in A: p_0=0\}$; it is easy to see that it is convex. Denote the ideal $\{\sum_{i=1}^{\infty} p_i y^i: p_i\in(x^2+1)\}$ by Q. Since $0\leqq y\leqq(x^2+1)y$, $(x^2+1)y\in Q$, but $y\notin Q$, the ideal Q is not convex. To see that Q is a prime ideal in A, assume that $\sum_{i=0}^{\infty} a_i y^i$, $\sum_{i=0}^{\infty} b_i y^i\notin Q$. Let a_n and b_m denote the first coefficients that fail to be in (x^2+1). Then, in the coefficient $\sum_{j=0}^{n+m} a_j b_{n+m-j}$ of the product, we have $a_j\in(x^2+1)$ for $j<n$ and $b_{n+m-j}\in(x^2+1)$ for $j>n$. Thus, each term except $a_n b_m$ belongs to (x^2+1). It follows that $\sum_{j=0}^{n+m} a_j b_{n+m-j}\notin(x^2+1)$, whence $(\sum_{i=0}^{\infty} a_i y^i)(\sum_{i=0}^{\infty} b_i y^i)\notin Q$.

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