ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THE WAVE EQUATIONS

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In the theory of scattering for hyperbolic equations, it is necessary to estimate the behavior of solutions to the unperturbed problem as well as the perturbed for large $|t|$. At present most estimates for the wave equation or the relativistic wave equation are in the sup norm. (See [1]–[5].) The purpose of this paper is to present some simple but rather interesting estimates in L_2 of solutions to

$$\Box u = m^2 u, \qquad m \geqq 0, \tag{1}$$

where

$$\Box = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \frac{\partial^2}{\partial t^2}.$$

In particular, we will show that for finite energy solutions u of (1) $\|u(x, t)\|_2$ has a definite limit depending on the initial data. It will follow that if $\|u(x, t)\|_2$ tends to 0, $u \equiv 0$. This seems to be a well-known "folk theorem."

1. **The L_2 norm.** Let $B = (m^2 - \Delta^2)^{1/2}$ considered as a linear operator on $L_2(R^n)$. If $m > 0$, B has a bounded inverse. Let $B(z) = (m^2 + z^2)^{1/2}$ where $z = (z_1, \cdots, z_n)$ and $z^2 = z_1^2 + \cdots + z_n^2$. We define the domain of B to be all $f \in L_2$ such that $B(z)F(z) \in L_2$ where F is the Fourier transform of f. For suitable initial data, the following two integrals are constant.

$$\text{(2)}\quad \Pi = \int_{R_n} \left\{ \sum_{i=1}^{n} \left(\frac{\partial u(x, t)}{\partial x_i} \right)^2 + u_t^2 + m^2 u^2 \right\} dx = \int_{R_n} \left\{ (Bu)^2 + u_t^2 \right\} dx,$$

$$\text{(3)}\quad \hat{\Gamma} = \int_{R_n} \left\{ u^2 + (B^{-1} u_t)^2 \right\} dx \qquad (\text{if } m = 0,\ u_t(x, 0) \in \mathfrak{D}_{B^{-1}}).$$

THEOREM 1. *Let $u(x, 0) = f$, $u_t(x, 0) = g$.*

$$\text{(1)}\qquad \lim_{|t| \to \infty} \int \left\{ \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2 + m^2 u \right\} dx = \frac{\Pi}{2},$$

$$\text{(2)}\qquad \lim_{|t| \to \infty} \int_{R_n} u^2(x, t)\, dx = \frac{\Gamma}{2}.$$

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PROOF. We will prove statement (2). (1) is similar. We first note that

$$\Gamma = \Gamma(0) = \int_{R_n} f^2 + (B^{-1}g)^2 dx$$
$$= \int_{\hat{E}_n} |F|^2 + (m^2 + |\xi|^2)^{-1} |G|^2 d\xi$$

where F, G are the Fourier transforms of f, g respectively with respect to x. By the functional calculus,

$$U(z, t) = (\cos tB(z))F(z) + (B^{-1}(z) \sin tB(z))G(z).$$

But

$$\int_{R_n} u^2 dx = \int_{\hat{E}_n} |U|^2 dz$$
$$= \int_{\hat{E}_n} [\cos^2 tB |F|^2 + B^{-2} \sin^2 tB |G|^2 + B^{-1} \sin 2tB(F\bar{G} + \bar{F}G)]dz$$
$$= \frac{1}{2} \int_{\hat{E}_n} |F|^2 + B^{-2} |G|^2$$
$$+ \int_{\hat{E}_n} [\tfrac{1}{2} \cos 2tB(|F|^2 - |B^{-1}G|^2) + B^{-1} \sin 2tB(F\bar{G} + \bar{F}G)]d\hat{z}\xi.$$

The theorem will be proved if we show the second integral tends to zero. However, this follows by a trivial modification of the Riemann-Lebesgue lemma.

COROLLARY. *Let u be solution of $\square u = 0$ with $u(x, 0) \in L^2$ and $u_t(x, 0)$ in the domain of B^{-1}. Then if $\|u(\cdot, t)\|_{L_2}$ tends to zero as $|t| \to \infty$, then $u = 0$ for all t.*

PROOF. By the assumptions, $\Gamma = 0$. Thus $u = 0$.

REMARK. Theorem 1 seems to suggest an equipartition or virial law of some kind for the energies.

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