SPACES BETWEEN A PAIR OF REFLEXIVE LEBESGUE SPACES

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The purpose of this paper is to determine conditions on the rearrangement invariant space X which make the following implication valid.

If T is a bounded operator from L^p to itself for all p with $1<p<\infty$, then T is a bounded operator from X into itself.

In other words, we seek conditions under which X will be intermediate between L^p and L^q for some p, q satisfying $1<p<q<\infty$ where p, and q depend on X.

Rearrangement invariant spaces. Let $(\Sigma, \mathfrak{F}, \mu)$ be a totally σ-finite measure space, and let $\mathfrak{M}(\Sigma)$, $\mathcal{P}(\Sigma)$ denote respectively the class of measurable, and nonnegative measurable functions on Σ. Given $f\in\mathfrak{M}(\Sigma)$, let f^* be the nonincreasing rearrangement of f onto $R^+ = [0, \infty)$. That is, f^* is the nonincreasing, left-continuous function on R^+ such that, if m is Lebesgue measure on R^+, for all $y>0$,

$$m\{t \in R^+ : f^*(t) > y\} = \mu\{x \in \Sigma : |f(x)| > y\}.$$

Let $\sigma: \mathcal{P}(R^+)\to[0, \infty]$ satisfy the following five conditions for all u, v, $u_n\in\mathcal{P}(R^+)$, all measurable sets E with $m(E)<\infty$, characteristic function χ_E, and all constants $a\geqq 0$.

(i) $\sigma(u)=0\Leftrightarrow u=0$ a.e., $\sigma(u+v)\leqq\sigma(u)+\sigma(v)$, $\sigma(au)=a\sigma(u)$,

(ii) $u_n\uparrow u$ a.e. $\Rightarrow\sigma(u_n)\uparrow\sigma(u)$,

(iii) $\sigma(\chi_E)<\infty$,

(iv) $\exists A_E<\infty$, such that $\int_E u\, dm\leqq A_E\sigma(u)$,

(v) $\sigma(u)=\sigma(u^*)$.

Conditions (i)–(iv) define a length function (metric) in Luxemburg's terminology [6].

Let $X(\Sigma)=\{f\in\mathfrak{M}(\Sigma):\sigma(f^*)<\infty\}$. Identifying functions which differ by a null function, $X(\Sigma)$ is a Banach space with norm given by $\|f\|=\sigma(f^*)$, for $f\in X(\Sigma)$ (see [1]). X is called a rearrangement invariant space.

We denote by $[\mathfrak{X}, \mathfrak{Y}]$ the space of bounded linear operators with domain the Banach space $\mathfrak{X}$ and range in the Banach space $\mathfrak{Y}$. Also, $[\mathfrak{X}, \mathfrak{X}]$ is written $[\mathfrak{X}]$.

We define the group of operators E_s with $\mathfrak{D}(E_s)=\mathfrak{M}(R^+)$ by

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$$(E_s f)(t) = f(st), \qquad 0 < s < \infty, \qquad f \in \mathfrak{M}(R^+).$$

Let the norm of E_s as a mapping from $X(R^+)$ into itself be denoted $h(s) = h(s; X)$. It is immediate that $h(1) = 1$ and $h(st) \leqq h(s)h(t)$. Also, if

$$\alpha = \inf_{0<s<1} [-\log h(s)/\log s]$$

and

$$\beta = \sup_{s>1} [-\log h(s)/\log s],$$

then $0 \leqq \beta \leqq \alpha \leqq 1$, and, given $\epsilon > 0$, there exist $\delta(\epsilon) > 0$ and $R(\epsilon) < \infty$ such that

$$s^{-\alpha} \leqq h(s) \leqq s^{-\alpha-\epsilon} \qquad \text{for } 0 < s < \delta(\epsilon),$$
$$s^{-\beta} \leqq h(s) \leqq s^{-\beta+\epsilon} \qquad \text{for } R(\epsilon) < s < \infty.$$

(See [**1**, p. 80], or [**4**, p. 244].)

THEOREM 1. *Suppose that* $T \in [L^p(\Sigma)]$ *and* $T \in [L^q(\Sigma)]$, *with* $1 < p < q < \infty$. *Let X be a rearrangement invariant space, and suppose that* α *and* β *defined above satisfy* $\alpha p < 1$, $\beta q > 1$. *Then* $T \in [X(\Sigma)]$.

COROLLARY 1. *Suppose that* $T \in [L^p(\Sigma)]$ *for all p with* $1 < p < \infty$. *If* $sh(s) \to 0$, *as* $s \to 0+$, *and* $h(s) \to 0$, *as* $s \to \infty$, *then* $T \in [X(\Sigma)]$.

To prove Theorem 1, we use an inequality due to Calderón [**3**].

Define the Lorentz space $\Lambda(p)$ to be the rearrangement invariant space with norm given by

$$\|f\|_{\Lambda(p)} = \frac{1}{p} \int_0^\infty t^{1/p} f^*(t) \frac{dt}{t}, \qquad 1 < p < \infty,$$

and the space $M(p)$ with norm given by

$$\|f\|_{M(p)} = \sup_{t>0} t^{(1/p)-1} \int_0^t f^*(s)\,ds, \qquad 1 < p < \infty.$$

It is easy to check that

$$\|f\|_{M(p)} \leqq \|f\|_p \leqq \|f\|_{\Lambda(p)}$$

where $\|f\|_p$ denotes the L^p-norm of f. Hence, $\Lambda(p) \subset L^p \subset M(p)$ and $[L^p] \subset [\Lambda(p), M(p)]$.

For $1 < p < \infty$, we say an operator is of weak type p if $T \in [\Lambda(p), M(p)]$, and this coincides with the classical definition due to Marcinkiewicz.

The space $\Lambda(p)+\Lambda(q)$, which is mentioned in the following lemma, is the Banach space of functions of the form $f+g$, where $f\in\Lambda(p)$, $g\in\Lambda(q)$, and

$$\|f+g\| = \inf_{f_1,g_1} (\|f_1\|_{\Lambda(p)} + \|g_1\|_{\Lambda(q)}),$$

the infimum being taken over all $f_1\in\Lambda(p)$, $g_1\in\Lambda(q)$, with $f_1+g_1=f+g$.

LEMMA 1 (CALDERÓN [3]). *Suppose that $1<p<q<\infty$, and that T is of weak types p and q. Then, there is a constant $c=c(p, q; T)$ such that, for all $f\in\Lambda(p)+\Lambda(q)$,*

$$(Tf)^*(t) \leqq c\int_0^\infty \frac{d}{ds}\min[(s/t)^{1/p}, (s/t)^{1/q}]f^*(s)ds. \tag{1}$$

It can be verified that $f\in\Lambda(p)+\Lambda(q)$ if and only if

$$\int_0^\infty \min(t^{1/p}, t^{1/q})f^*(t)\frac{dt}{t} < \infty. \tag{2}$$

We shall find it convenient to rewrite (1) in the following way,

$$(Tf)^*(t) \leqq c'[(P_pf^*)(t) + (Q_qf^*)(t)], \tag{3}$$

where the operators P_p and Q_q are defined by

$$(P_pf)(t) = \int_0^1 f(st)s^{(1/p)-1}ds,$$

and

$$(Q_qf)(t) = \int_1^\infty f(st)s^{(1/q)-1}ds.$$

The domains of P_p and Q_q consist of all $f\in\mathcal{P}(R^+)$ for which the required integrals exist a.e.

LEMMA 2 (SEE [2]). *If $(Af)(t)=\int_0^\infty a(s)f(st)ds$, with*

$$\mathfrak{D}(A) = \left\{f\in\mathfrak{M}(R^+): \int_0^\infty |a(s)|\cdot|f(st)|\,ds < \infty \text{ a.e.}\right\}$$

and if $\|A\|$ denotes the norm of A as a member of $[X(R^+)]$, then

$$\|A\| \leqq \int_0^\infty |a(s)|\cdot h(s; X)\,ds.$$

Proof of Theorem 1. Suppose that $T\in[L^p]$, $T\in[L^q]$, $1<p<q<\infty$, then T is of weak type p and q; so (3) holds for all $f\in\Lambda(p)+\Lambda(q)$. We assume that $\alpha p<1$, and $\beta q>1$, and show that this implies that $P_p\in[X(R^+)]$, $Q_q\in[X(R^+)]$ and also that $X(\Sigma)\subset\Lambda(p)+\Lambda(q)$.

For, if $\alpha p<1$, we can choose $\epsilon>0$ so that $\alpha+\epsilon<1/p$, and then

$$\int_0^{\delta(\epsilon)} s^{(1/p)-1}h(s)ds \leqq \int_0^{\delta(\epsilon)} s^{-\alpha-\epsilon-1+(1/p)}ds < \infty.$$

Since $h(s)$ is bounded in $[\delta(\epsilon), 1]$, we have

$$\int_{\delta(\epsilon)}^1 s^{(1/p)-1}h(s)\, ds < \infty$$

so that $P_p\in[X(R^+)]$ by Lemma 2.

Similarly, $\beta q>1$ implies $Q_q\in[X(R^+)]$.

If $f\in X(\Sigma)$, then $f^*\in X(R^+)$, so that $(P_pf^*)(t)<\infty$ a.e., and $(Q_qf^*)(t)<\infty$ a.e. But, P_pf^* and Q_qf^* are nonincreasing, so $(P_pf^*)(1)<\infty$ and $(Q_qf^*)(1)<\infty$, and hence

$$\int_0^\infty \min\,(s^{1/p}, s^{1/q})f^*(s)\,\frac{ds}{s} = (P_pf^*)(1) + (Q_qf^*)(1) < \infty.$$

Thus, by (2), $f\in\Lambda(p)+\Lambda(q)$, so $X(\Sigma)\subset\Lambda(p)+\Lambda(q)$. Hence (3) applies for all $f\in X(\Sigma)$ so that, if $\sigma(\cdot)$ denotes the norm in $X(R^+)$ and; $\|\cdot\|$ the norm in $X(\Sigma)$, then

$$\begin{aligned}\|Tf\| = \sigma((Tf)^*) &\leqq c'\{\sigma(P_pf^*) + \sigma(Q_qf^*)\}, \text{ from (3)},\\ &\leqq c'(a+b)\sigma(f^*) = c''\|f\|,\end{aligned}$$

where $a=\|P_p\|$ and $b=\|Q_q\|$ as members of $[X(R^+)]$.

Proof of Corollary 1. If $sh(s)\to 0$, as $s\to 0+$, then $\alpha<1$; so for some $p>1$, we have $p\alpha<1$. Similarly, if $h(s)\to 0$, as $s\to\infty$, then $\beta>0$ so that, for some $q<\infty$, $\beta q>1$. Hence, if $T\in[L^r]$ for all r with $1<r<\infty$, then in particular $T\in[L^p]$ and $T\in[L^q]$. Thus we can apply Theorem 1 to show that $T\in[X(\Sigma)]$.

Corollary 2. *Suppose that $T\in[L^p]$ for all p with $1<p<\infty$. Suppose that Σ is nonatomic and $\mu(\Sigma)=\infty$. If $L_\Phi(\Sigma)$ is a reflexive Orlicz space, then $T\in[L_\Phi(\Sigma)]$.*

Proof. By Lemmas 5.9 and 5.11 of [2], $L_\Phi(\Sigma)$ is reflexive if and only if $sh(s)\to 0$ as $s\to 0+$, and $h(s)\to 0$ as $s\to\infty$.

Remarks. In case Σ is one of R^+, R, R^n, Corollary 1 is a best possible result in the sense that there is an operator $T\in[L^p(\Sigma)]$ for all

p with $1<p<\infty$, and such that $T\in[X(\Sigma)]$ if and only if $sh(s)\to 0$ as $s\to 0+$, and $h(s)\to 0$ as $s\to\infty$. For R^+, T can be taken to be the Stieltjes transform; for R, the Hilbert transform; and for R^n, any of the M. Riesz singular integral transforms. (See [2] for the first two of these results.)

Corollary 2 complements an analogous result of Ryan which corresponds to the case $\mu(\Sigma)<\infty$ (see [7]).

The proof of Theorem 1 shows that the hypotheses need only be that T is of weak types p and q for the conclusion to be valid.

References

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