ON REAL PROJECTIVE SPACES AS FINSLER MANIFOLDS

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Let M be a finite-dimensional C^3 manifold supplied with a C^2 Finsler metric $ds=F(x, dx)$, which is not necessarily even in dx. Let ρ designate the induced oriented topological metric. For any $p\in M$, the *antipodal locus* of p is the set $A(p)=\{q\in M \mid \rho(p, q)\geqq\rho(p, r)$ for all $r\in M\}$. For example, if M is a real projective space with the Riemannian metric of constant curvature 1, $A(p)$ is a smooth hypersurface (in fact, a projective hyperplane) for every $p\in M$. One may ask, how close does this property come to characterizing real projective spaces among Finsler manifolds?

We prove

THEOREM. *Let M be a finite-dimensional C^3 manifold supplied with a complete C^2 Finsler metric. Suppose $A(p)$ is a smooth hypersurface for at least one $p\in M$. Then M is homeomorphic with a real projective space.*

PROOF. All the tools for the proof are contained in the classic paper of J. H. C. Whitehead [1], especially in §10.

Let p be a point for which $A(p)$ is a smooth hypersurface. Let exp denote the exponential map (which is of class C^1) of the Finsler metric at p. Let δ be the distance from p to $A(p)$ and let $S(\delta) =\{v\in M_p \mid F(p, v)=\delta\}$. $S(\delta)$ is homeomorphic with a sphere and bounds a cell.

$A(p)$ is a subset of the cut locus of p and hence any $q\in A(p)$ is either hit by at least two minimal geodesics from p to q or is conjugate to p or maybe both. In the first case, suppose $q=\exp v=\exp w$, $v\neq w$, and that q is not conjugate to p. If also $q=\exp u$, $u\neq v, w$, q would be a branch point for $A(p)$ contrary to $A(p)$ being a manifold. The second case cannot occur, since if only one geodesic of length δ went from p to q, the conjugacy of q would imply that $A(p)$ had codimension at least 2 at p. In the third case, Whitehead shows that the intersection of the cut and conjugate loci is again of codimension at least 2.

Now, let $A'(p)\subset S(\delta)$ be the subset of those elements carried by exp onto $A(p)$. Since $A(p)$ is closed and exp is continuous, $A'(p)$ is closed in $S(\delta)$. Since exp is nonsingular on $A'(p)$ and the dimensions of $S(\delta)$ and $A(p)$ coincide, $A'(p)$ is open in $S(\delta)$. Therefore, $A'(p)=S(\delta)$.

Received by the editors March 22, 1966.

The interior of $S(\delta)$ is mapped by exp diffeomorphically onto $M-A(p)$ and we have just shown that $S(\delta)$ is a 2–1 covering of $A(p)$. The homeomorphism of M with a real projective space is now obvious.

Reference

1. J. H. C. Whitehead, *On the covering of a complete space by the geodesics through a point*, Ann. of Math. **36** (1935), 679–704.

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