1. Introduction. Let $H_k$ be the space of polynomials of degree not greater than $2k + 1$, and let $f = p + w$, where $p$ is in $H_k$ and $w$ is a sample function from a stationary random process with zero mean and spectral density $\psi(\theta)$. We study the problem of filtering $f$ to obtain estimates of $p$ and its derivatives. Consider the polynomial weighting functions

$$u_{ik}(x) = \sum_{j=0}^{2k+1} (j + \frac{1}{2}) P_j^{(2i)}(0) P_j(x),$$

$$v_{ik}(x) = \sum_{j=0}^{2k+1} (j + \frac{1}{2}) P_j^{(2i+1)}(0) P_j(x), \quad 0 \leq i \leq n,$$

where $P_0, P_1, \ldots$ are the Legendre polynomials. Define

$$g_i(x) = \int_{-1}^{1} u_{ik}(y)f(x + y)dy$$

and

$$h_i(x) = \int_{-1}^{1} v_{ik}(y)f(x + y)dy.$$ (1)

The mean value of $g_i(x)$ is given by

$$E[g_i(x)] = \int_{-1}^{1} u_{ik}(y)p(x + y)dy = p^{(2i)}(x),$$ (2)

and its random component is

$$\mu_i(x) = \int_{-1}^{1} u_{ik}(y)w(x + y)dy.$$ 

The second equality in (2) follows from the well-known reproducing property of the kernel

$$K_k(x, y) = \sum_{j=0}^{2k+1} (j + \frac{1}{2}) P_j(x)P_j(y)$$

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in \( H_k \). (More generally, it is well known that if \( F \) is square integrable in \([-1, 1]\), then
\[
\int_{-1}^{1} K_k(x, y)F(y)\,dy
\]
is the least squares polynomial fit to \( F \) of degree not greater than \( 2k+1 \).) It can be shown that \( r_i(x) \) is a sample function from a stationary random process with zero mean and spectral density

\[
\alpha_i(\theta) = C_{ik}(\theta)\psi(\theta),
\]
where

\[
C_{ik}(\theta) = \int_{-1}^{1} u_{ik}(x) \cos x\,dx.
\]

It is also known that if the process \( w \) is white noise, (that is, if \( \psi = \text{constant} \), then \( g_i(x) \) is the minimum variance estimate of \( \rho^{(2i)}(x) \). Similarly,

\[
E[h_i(x)] = \rho^{(2i+1)}(x),
\]
and the error

\[
\rho^{(2i+1)}(x) - h_i(x)
\]
has zero mean and spectral density

\[
\beta_i(\theta) = S_{ik}(\theta)\psi(\theta),
\]
where

\[
S_{ik}(\theta) = \int_{-1}^{1} v_{ik}(x) \sin x\,dx.
\]

The magnitudes of \( C_{ik} \) and \( S_{ik} \) are of interest because of (3) and (6). In [1], the author established the following theorem.

**Theorem 1.** For each \( k \geq 0 \), \( C_{0k}(0) = 1 \), and \( |C_{0k}(\theta)| < 1 \) if \( \theta \neq 0 \).

In this paper we prove the following theorem.

**Theorem 2.** Let \( k \geq 0 \), and \( 0 \leq i \leq k \). Then

\[
|\theta^{-2i}C_{ik}(\theta)| < 1,
\]
and

\[
|\theta^{-(2i+1)}S_{ik}(\theta)| < 1,
\]
if \( \theta \neq 0 \), and

\[
\lim_{\theta \to 0} \theta^{-2i} C_{ik}(\theta) = \lim_{\theta \to 0} \theta^{-(2i+1)} S_{ik}(\theta) = (-1)^i.
\]

2. **Proof of Theorem 2.** The proof uses Theorem 1 and an induction argument. We need certain lemmas.

**Lemma 1.** Let

\[
F_i(\theta) = \int_{-1}^{1} x^{2i} \cos x \, dx,
\]
\[
G_i(\theta) = \int_{-1}^{1} x^{2i+1} \sin x \, dx,
\]

and

\[
\sigma_i = \int_{-1}^{1} x^{2i} \, dx = (i + \frac{1}{2})^{-1}.
\]

Then

\[
F_i(\theta) = \sum_{j=0}^{k} \sigma_{i+j} \frac{C_{jk}(\theta)}{(2j)!}, \quad 0 \leq i \leq k,
\]

and

\[
G_i(\theta) = \sum_{j=0}^{k} \sigma_{i+j+1} \frac{S_{jk}(\theta)}{(2j + 1)!}, \quad 0 \leq i \leq k.
\]

**Proof.** Let

\[
f_k(x, \theta) = \int_{-1}^{1} K_k(x, y) \cos y \theta dy.
\]

Since \( f_k \) is the projection of \( \cos x \theta \) on \( H_k \),

\[
\int_{-1}^{1} [\cos y \theta - f_k(y, \theta)] y^{2i} \, dy = 0, \quad 0 \leq i \leq k.
\]

By expanding \( K_k \) in powers of \( x \), we find that

\[
f_k(x, \theta) = \sum_{j=0}^{k} \frac{C_{jk}(\theta)x^{2j}}{(2j)!},
\]

which, when substituted into (14), yields (12). Equation (13) can be found similarly, by considering the projection of \( \sin x \theta \) on \( H_k \).
Lemma 2. If \( C_{kk}(\alpha) = 0 \), then \( C_{ik}(\alpha) = C_{i,k-1}(\alpha) \), \( 0 \leq i \leq k-1 \), and if \( S_{kk}(\beta) = 0 \), then \( S_{ik}(\beta) = S_{i,k-1}(\beta) \), \( 0 \leq i \leq k-1 \).

Proof. If the highest degree term in \( f_k \) vanishes, then \( f_k = f_{k-1} \), from elementary properties of least squares polynomials. This implies the first part, and the second is obtained similarly.

Lemma 3. Let

\[
\alpha_{ik} = \left( (-1)^i \prod_{q=0}^{k} (i + q + \frac{1}{2}) \right) / i! (k - i)!
\]

and

\[
\beta_{ik} = \left( (-1)^i \prod_{q=0}^{k} (i + q + \frac{3}{2}) \right) / i! (k - i)!.
\]

Then

\[
C_{ik}(\theta) = (2i)! \alpha_{ik} \sum_{j=0}^{k} \alpha_{jk} \sigma_{i+j} F_j(\theta),
\]

and

\[
S_{ik}(\theta) = (2i + 1)! \beta_{ik} \sum_{j=0}^{k} \beta_{jk} \sigma_{i+j+1} G_j(\theta).
\]

Proof. The matrices of the systems (12) and (13) have general elements of the form \((a_i + b_j)^{-1}\). Using the method of [2, Vol. 2, pp. 98, 299], it can be shown that their inverses have \(i,j\)th elements given by \(\alpha_{ik} \alpha_{jk} \sigma_{i+j}\) and \(\beta_{ik} \beta_{jk} \sigma_{i+j+1}\), respectively, where \(0 \leq i, j \leq k\).

Lemma 4.

\[
\frac{d}{d\theta} \{ \theta^{-2i} S_{i,k-1}(\theta) \} = - \frac{\beta_{i,k-1}(2i + 1)!}{\alpha_{kk}(2k)!} C_{kk}(\theta) \theta^{-2i-1},
\]

(16)

\[
0 \leq i \leq k - 1,
\]

and

\[
\frac{d}{d\theta} \{ \theta^{-2i} C_{ik}(\theta) \} = - \frac{\alpha_{ik}(2i)!}{\beta_{kk}(2k + 1)!} S_{kk}(\theta) \theta^{-2i},
\]

(17)

\[
0 \leq i \leq k.
\]

Proof. To establish (16), we start by dividing (15), with \( k \) replaced by \( k - 1 \), by \( \theta^{2i+1} \), and differentiating. Note that
\[
\frac{d}{d\theta} \left\{ G_j(\theta) \theta^{-2i-1} \right\} = G_j'(\theta) \theta^{-2i-1} - (2i + 1)G_j(\theta) \theta^{-2i-2}
\]
\[
= \theta^{-2i-1} \left( \frac{2i + 2j + 3}{2j + 2} F_{j+1}(\theta) - \frac{2i + 1}{j + 1} \sin \theta \right),
\]

where the last equality follows from the relations

(18) \[ G_j'(\theta) = F_{j+1}(\theta), \]

and

\[
G_j(\theta) = \frac{\sin \theta}{(j + 1)} - \frac{(\theta F_{j+1}(\theta))/(2j + 2)}{(2j + 2)},
\]

which can be deduced from (10) and (11). Using the fact that

\[ F_0(\theta) = \frac{2 \sin \theta}{\theta}, \]

it can now be seen that

\[
\frac{d}{d\theta} \left( \theta^{-2i-1} S_{i,k-1}(\theta) \right)
\]
\[
= \left( \frac{2i + 1}{\theta^{2i+1}} \sum_{j=0}^{k-1} \frac{\beta_{j,k-1} F_{j+1}(\theta)}{j + 1} - \frac{F_0(\theta)}{\sigma_i} \sum_{j=0}^{k-1} \frac{\beta_{j,k-1} \sigma_{i+j+1}}{j + 1} \right).
\]

It is easy to verify that

\[ \beta_{j,k-1}/(j + 1) = - \alpha_{j+1,k} \sigma_{j+k+1} \]

and

\[ \sum_{j=0}^{k} \alpha_{jk} \sigma_{j+k} \sigma_{i+j} = 0, \quad 0 \leq i \leq k - 1. \]

Using these and (19), (16) can be established. Equation (17) can be derived similarly, using the relations

\[ F_j'(\theta) = -G_j(\theta), \quad F_j(\theta) = \sigma_j \left( \cos \theta + \frac{\theta}{2} G_j(\theta) \right), \]

(20) \[ \alpha_{jk} \sigma_j = \beta_{jk} \sigma_{k+j+1}, \]

and

\[ \sum_{j=0}^{k} \alpha_{jk} \sigma_{i+j} \sigma_j = 0, \quad 1 \leq i \leq k. \]

**Lemma 5.** The functions \( \theta^{-2i} C_{ik}(\theta), 0 \leq i \leq k, \) have the same critical points. Also, the functions \( \theta^{-2i-1} S_{ik}(\theta), 0 \leq i \leq k, \) have the same critical points.
points. Furthermore, if $\phi$ is a critical point of the former, then
\begin{equation}
-C_{i, k}(\phi)/\phi = S_{i-1, k-1}(\phi), \quad 1 \leq i \leq k,
\end{equation}
and if $\eta$ is a critical point of the latter, then
\begin{equation}
\frac{S_{i, k}(\eta)}{\eta} = C_{i, k}(\eta), \quad 0 \leq i \leq k.
\end{equation}

**Proof.** From Lemma 4, the critical points of the first set of functions are the zeroes of $S_{kk}$, while the critical points of the second set are the zeroes of $C_{k+1, k+1}$. This implies the first two statements. To prove (21), multiply and divide the $j$th term on the right of (12) by $\theta^{2i}$ and differentiate, to obtain
\begin{equation}
F_i'(\theta) = \sum_{j=0}^{k} \sigma_{i+j} \frac{C_{j, k}(\theta)}{(2j)! \theta} + \sum_{j=0}^{k} \sigma_{i+j} \frac{\theta^{2i}}{(2j)!} \frac{d}{d\theta} (\theta^{-2i}C_{j, k}(\theta)).
\end{equation}

Let $\theta = \phi$, where $S_{kk}(\phi) = 0$, so that the second sum vanishes. Making use of (20), we can write
\begin{equation}
G_i(\phi) = - \sum_{j=0}^{k-1} \sigma_{i+j} \frac{C_{j+1, k}(\phi)}{(2j+1)!} \phi, \quad 0 \leq i \leq k.
\end{equation}

Since $S_{kk}(\phi) = 0$, we can infer (21) by comparing this system with (13) (with $\theta = \phi$), noting that the solution of the latter is unique, and using Lemma 2.

To derive (22), let $C_{k+1, k+1}(\eta) = 0$. Use (12) with $k$ replaced by $k+1$ and $\theta = \eta$, and Lemma 2, to conclude that
\begin{equation}
F_i(\eta) = \sum_{j=0}^{k} \frac{C_{j, k}(\eta)}{(2j)!}, \quad 0 \leq i \leq k+1.
\end{equation}
The last $k$ of these equations can be written
\begin{equation}
F_{i+1}(\eta) = \sum_{j=0}^{k} \sigma_{i+j+1} \frac{C_{j+1, k}(\eta)}{(2j)!}, \quad 0 \leq i \leq k.
\end{equation}
Now multiply and divide the $j$th term on the right side of (13) by $\theta^{2i+1}$, differentiate as in the derivation of (23), set $\theta = \eta$, and use (18). The result is
\begin{equation}
F_{i+1}(\eta) = \sum_{j=0}^{k} \sigma_{i+j+1} \frac{S_{j, k}(\eta)}{(2j)!}, \quad 0 \leq i \leq k.
\end{equation}
Comparing this with (24) yields (22).
We can now complete the proof of Theorem 2, by induction. Assume that \( k \geq 1 \), and that Theorem 2 holds for \( k-1 \). This is so if \( k = 1 \), from Theorem 1. Let \( \phi \) be a nonzero critical point of \( \theta^{-2i}C_{ik}(\theta) \), and divide both sides of (21) by \( \phi^{2i-1} \) to obtain

\[
-\phi^{-2i}C_{ik}(\phi) = \phi^{-2i+1}S_{i-1,k-1}(\phi), \quad 1 \leq i \leq k.
\]

From the induction assumption, it now follows that

\[
|\theta^{-2i}C_{ik}(\theta)| < 1, \quad 1 \leq i \leq k,
\]

if \( \theta = \phi \), a nonzero critical point of \( \theta^{-2i}C_{ik}(\theta) \), and therefore the inequality holds for all \( \theta \neq 0 \). (The inequality with \( i = 0 \) does not follow from this argument, but it has already been established in Theorem 1.) Now let \( \eta \) be a nonzero critical point of \( \theta^{-2i-1}S_{ik}(\theta) \), and divide both sides of (22) by \( \eta^{2i} \) to find that

\[
\eta^{-2i-1}S_{ik}(\eta) = \eta^{-2i}C_{ik}(\eta), \quad 0 \leq i \leq k.
\]

We have just established that the right side is less than unity in magnitude. Hence (8) holds for \( \theta = \eta \), and therefore for any \( \theta \neq 0 \).

The limits (9) are obtained by expanding (4) and (7) in powers of \( \theta \), integrating term by term, and noting that

\[
\int_{-1}^{1} u_{ik}(x)x^{2i}dx = (2i)!\delta_{ij},
\]

\[
\int_{-1}^{1} v_{ik}(x)x^{2i+1}dx = (2i + 1)!\delta_{ij}, \quad 0 \leq i, j \leq k.
\]

References


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