BOUNDS ON THE GENERATING FUNCTIONS OF CERTAIN SMOOTHING OPERATIONS

WILLIAM F. TRENCH

1. **Introduction.** Let H_k be the space of polynomials of degree not greater than $2k+1$, and let $f=p+w$, where p is in H_k and w is a sample function from a stationary random process with zero mean and spectral density $\psi(\theta)$. We study the problem of filtering f to obtain estimates of p and its derivatives. Consider the polynomial weighting functions

$$u_{ik}(x) = \sum_{j=0}^{2k+1} (j+\tfrac{1}{2})P_j^{(2i)}(0)P_j(x),$$

$$v_{ik}(x) = \sum_{j=0}^{2k+1} (j+\tfrac{1}{2})P_j^{(2i+1)}(0)P_j(x), \qquad 0 \leqq i \leqq n,$$

where $P_0, P_1, \cdots$ are the Legendre polynomials. Define

$$g_i(x) = \int_{-1}^{1} u_{ik}(y)f(x+y)dy$$

and

$$h_i(x) = \int_{-1}^{1} v_{ik}(y)f(x+y)dy. \tag{1}$$

The mean value of $g_i(x)$ is given by

$$E[g_i(x)] = \int_{-1}^{1} u_{ik}(y)p(x+y)dy = p^{(2i)}(x), \tag{2}$$

and its random component is

$$\mu_i(x) = \int_{-1}^{1} u_{ik}(y)w(x+y)dy.$$

The second equality in (2) follows from the well-known reproducing property of the kernel

$$K_k(x, y) = \sum_{j=0}^{2k+1} (j+\tfrac{1}{2})P_j(x)P_j(y)$$

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in H_k. (More generally, it is well known that if F is square integrable in $[-1, 1]$, then

$$\int_{-1}^{1} K_k(x, y)F(y)dy$$

is the least squares polynomial fit to F of degree not greater than $2k+1$.) It can be shown that $r_i(x)$ is a sample function from a stationary random process with zero mean and spectral density

$$\alpha_i(\theta) = C_{ik}^2(\theta)\psi(\theta), \tag{3}$$

where

$$C_{ik}(\theta) = \int_{-1}^{1} u_{ik}(x) \cos x\theta dx. \tag{4}$$

It is also known that if the process w is white noise, (that is, if $\psi=\text{constant}$), then $g_i(x)$ is the minimum variance estimate of $p^{(2i)}(x)$. Similarly,

$$E[h_i(x)] = p^{(2i+1)}(x), \tag{5}$$

and the error

$$p^{(2i+1)}(x) - h_i(x)$$

has zero mean and spectral density

$$\beta_i(\theta) = S_{ik}^2(\theta)\psi(\theta), \tag{6}$$

where

$$S_{ik}(\theta) = \int_{-1}^{1} v_{ik}(x) \sin x\theta dx. \tag{7}$$

The magnitudes of C_{ik} and S_{ik} are of interest because of (3) and (6). In [1], the author established the following theorem.

THEOREM 1. *For each* $k\geq 0$, $C_{0k}(0)=1$, *and* $|C_{0k}(\theta)|<1$ *if* $\theta\neq 0$.

In this paper we prove the following theorem.

THEOREM 2. *Let* $k\geq 0$, *and* $0\leq i\leq k$. *Then*

$$|\theta^{-2i}C_{ik}(\theta)| < 1,$$

and

$$|\theta^{-(2i+1)}S_{ik}(\theta)| < 1, \tag{8}$$

if $\theta \neq 0$, *and*

(9) $$\lim_{\theta\to 0} \theta^{-2i} C_{ik}(\theta) = \lim_{\theta\to 0} \theta^{-(2i+1)} S_{ik}(\theta) = (-1)^i.$$

2. **Proof of Theorem 2.** The proof uses Theorem 1 and an induction argument. We need certain lemmas.

LEMMA 1. *Let*

(10) $$F_i(\theta) = \int_{-1}^{1} x^{2i} \cos x\theta dx,$$

(11) $$G_i(\theta) = \int_{-1}^{1} x^{2i+1} \sin x\theta dx,$$

and

$$\sigma_i = \int_{-1}^{1} x^{2i} dx = (i + \tfrac{1}{2})^{-1}.$$

Then

(12) $$F_i(\theta) = \sum_{j=0}^{k} \sigma_{i+j} \frac{C_{jk}(\theta)}{(2j)!}, \qquad 0 \leqq i \leqq k,$$

and

(13) $$G_i(\theta) = \sum_{j=0}^{k} \sigma_{i+j+1} \frac{S_{jk}(\theta)}{(2j+1)!}, \qquad 0 \leqq i \leqq k.$$

PROOF. Let

$$f_k(x, \theta) = \int_{-1}^{1} K_k(x, y) \cos y\theta dy.$$

Since f_k is the projection of $\cos x\theta$ on H_k,

(14) $$\int_{-1}^{1} [\cos y\theta - f_k(y, \theta)] y^{2i} dy = 0, \qquad 0 \leqq i \leqq k.$$

By expanding K_k in powers of x, we find that

$$f_k(x, \theta) = \sum_{j=0}^{k} \frac{C_{jk}(\theta) x^{2j}}{(2j)!},$$

which, when substituted into (14), yields (12). Equation (13) can be found similarly, by considering the projection of $\sin x\theta$ on H_k.

LEMMA 2. *If* $C_{kk}(\alpha)=0$, *then* $C_{ik}(\alpha)=C_{i,k-1}(\alpha)$, $0\leqq i\leqq k-1$, *and if* $S_{kk}(\beta)=0$, *then* $S_{ik}(\beta)=S_{i,k-1}(\beta)$, $0\leqq i\leqq k-1$.

PROOF. If the highest degree term in f_k vanishes, then $f_k=f_{k-1}$, from elementary properties of least squares polynomials. This implies the first part, and the second is obtained similarly.

LEMMA 3. *Let*

$$\alpha_{ik} = \left((-1)^i \prod_{q=0}^{k} (i+q+\tfrac{1}{2})\right)/i!(k-i)!$$

and

$$\beta_{ik} = \left((-1)^i \prod_{q=0}^{k} (i+q+\tfrac{3}{2})\right)/i!(k-i)!.$$

Then

$$C_{ik}(\theta) = (2i)!\alpha_{ik} \sum_{j=0}^{k} \alpha_{jk}\sigma_{i+j}F_j(\theta),$$

and

$$S_{ik}(\theta) = (2i+1)!\beta_{ik} \sum_{j=0}^{k} \beta_{jk}\sigma_{i+j+1}G_j(\theta). \tag{15}$$

PROOF. The matrices of the systems (12) and (13) have general elements of the form $(a_i+b_j)^{-1}$. Using the method of [2, Vol. 2, pp. 98, 299], it can be shown that their inverses have i,jth elements given by $\alpha_{ik}\alpha_{jk}\sigma_{i+j}$ and $\beta_{ik}\beta_{jk}\sigma_{i+j+1}$, respectively, where $0\leqq i,\ j\leqq k$.

LEMMA 4.

$$\frac{d}{d\theta}\left\{\theta^{-2i-1}S_{i,k-1}(\theta)\right\} = -\frac{\beta_{i,k-1}(2i+1)!}{\alpha_{kk}(2k)!}C_{kk}(\theta)\theta^{-2i-1}, \quad 0\leqq i\leqq k-1, \tag{16}$$

and

$$\frac{d}{d\theta}\left\{\theta^{-2i}C_{ik}(\theta)\right\} = -\frac{\alpha_{ik}(2i)!}{\beta_{kk}(2k+1)!}S_{kk}(\theta)\theta^{-2i}, \qquad 0\leqq i\leqq k. \tag{17}$$

PROOF. To establish (16), we start by dividing (15), with k replaced by $k-1$, by θ^{2i+1}, and differentiating. Note that

$$\frac{d}{d\theta}\{G_j(\theta)\theta^{-2i-1}\} = G_j'(\theta)\theta^{-2i-1} - (2i+1)G_j(\theta)\theta^{-2i-2}$$

$$= \theta^{-2i-1}\left(\frac{2i+2j+3}{2j+2}F_{j+1}(\theta) - \frac{2i+1}{j+1}\frac{\sin\theta}{\theta}\right),$$

where the last equality follows from the relations

$$G_j'(\theta) = F_{j+1}(\theta), \tag{18}$$

and

$$G_j(\theta) = (\sin\theta)/(j+1) - (\theta F_{j+1}(\theta))/(2j+2),$$

which can be deduced from (10) and (11). Using the fact that

$$F_0(\theta) = (2\sin\theta)/\theta,$$

it can now be seen that

$$\begin{aligned} &\frac{d}{d\theta}(\theta^{-2i-1}S_{i,k-1}(\theta)) \\ &= \frac{(2i+1)!\beta_{i,k-1}}{\theta^{2i+1}}\left\{\sum_{j=0}^{k-1}\frac{\beta_{j,k-1}F_{j+1}(\theta)}{j+1} - \frac{F_0(\theta)}{\sigma_i}\sum_{j=0}^{k-1}\frac{\beta_{j,k-1}\sigma_{i+j+1}}{j+1}\right\}. \end{aligned} \tag{19}$$

It is easy to verify that

$$\beta_{j,k-1}/(j+1) = -\alpha_{j+1,k}\sigma_{j+k+1}$$

and

$$\sum_{j=0}^{k}\alpha_{jk}\sigma_{j+k}\sigma_{i+j} = 0, \qquad 0 \leqq i \leqq k-1.$$

Using these and (19), (16) can be established. Equation (17) can be derived similarly, using the relations

$$\begin{aligned} &F_j'(\theta) = -G_j(\theta), \qquad F_j(\theta) = \sigma_j\left(\cos\theta + \frac{\theta}{2}G_j(\theta)\right), \\ &\alpha_{jk}\sigma_j = \beta_{jk}\sigma_{k+j+1}, \end{aligned} \tag{20}$$

and

$$\sum_{j=0}^{k}\alpha_{jk}\sigma_{i+j}\sigma_j = 0, \qquad 1 \leqq i \leqq k.$$

LEMMA 5. *The functions* $\theta^{-2i}C_{ik}(\theta)$, $0\leqq i\leqq k$, *have the same critical points. Also, the functions* $\theta^{-2i-1}S_{ik}(\theta)$, $0\leqq i\leqq k$, *have the same critical* *points. Furthermore, if ϕ is a critical point of the former, then*

$$-C_{ik}(\phi)/\phi = S_{i-1,k-1}(\phi), \qquad 1 \leqq i \leqq k, \tag{21}$$

and if η is a critical point of the latter, then

$$\frac{S_{ik}(\eta)}{\eta} = C_{ik}(\eta), \qquad 0 \leqq i \leqq k. \tag{22}$$

PROOF. From Lemma 4, the critical points of the first set of functions are the zeroes of S_{kk}, while the critical points of the second set are the zeroes of $C_{k+1,k+1}$. This implies the first two statements. To prove (21), multiply and divide the jth term on the right of (12) by θ^{2j} and differentiate, to obtain

$$F_i'(\theta) = \sum_{j=1}^{k} \sigma_{i+j} \frac{C_{jk}(\theta)}{(2j-1)!\theta} + \sum_{j=0}^{k} \sigma_{i+j} \frac{\theta^{2j}}{(2j)!} \frac{d}{d\theta} (\theta^{-2j} C_{jk}(\theta)). \tag{23}$$

Let $\theta=\phi$, where $S_{kk}(\phi)=0$, so that the second sum vanishes. Making use of (20), we can write

$$G_i(\phi) = -\sum_{j=0}^{k-1} \sigma_{i+j+1} \frac{C_{j+1,k}(\phi)}{(2j+1)!\phi}, \qquad 0 \leqq i \leqq k.$$

Since $S_{kk}(\phi)=0$, we can infer (21) by comparing this system with (13) (with $\theta=\phi$), noting that the solution of the latter is unique, and using Lemma 2.

To derive (22), let $C_{k+1,k+1}(\eta)=0$. Use (12) with k replaced by $k+1$ and $\theta=\eta$, and Lemma 2, to conclude that

$$F_i(\eta) = \sum_{j=0}^{k} \sigma_{i+j} \frac{C_{jk}(\eta)}{(2j)!}, \qquad 0 \leqq i \leqq k+1.$$

The last k of these equations can be written

$$F_{i+1}(\eta) = \sum_{j=0}^{k} \sigma_{i+j+1} \frac{C_{jk}(\eta)}{(2j)!}, \qquad 0 \leqq i \leqq k. \tag{24}$$

Now multiply and divide the jth term on the right side of (13) by θ^{2j+1}, differentiate as in the derivation of (23), set $\theta=\eta$, and use (18). The result is

$$F_{i+1}(\eta) = \sum_{j=0}^{k} \sigma_{i+j+1} \frac{S_{jk}(\eta)}{(2j)!\eta}, \qquad 0 \leqq i \leqq k.$$

Comparing this with (24) yields (22).

We can now complete the proof of Theorem 2, by induction. Assume that $k \geqq 1$, and that Theorem 2 holds for $k-1$. This is so if $k=1$, from Theorem 1. Let ϕ be a nonzero critical point of $\theta^{-2i}C_{ik}(\theta)$, and divide both sides of (21) by ϕ^{2i-1} to obtain

$$-\phi^{-2i}C_{ik}(\phi) = \phi^{-2i+1}S_{i-1,k-1}(\phi), \qquad 1 \leqq i \leqq k.$$

From the induction assumption, it now follows that

$$\left| \theta^{-2i}C_{ik}(\theta) \right| < 1, \qquad 1 \leqq i \leqq k,$$

if $\theta=\phi$, a nonzero critical point of $\theta^{-2i}C_{ik}(\theta)$, and therefore the inequality holds for all $\theta \neq 0$. (The inequality with $i=0$ does not follow from this argument, but it has already been established in Theorem 1.) Now let η be a nonzero critical point of $\theta^{-2i-1}S_{ik}(\theta)$, and divide both sides of (22) by η^{2i} to find that

$$\eta^{-2i-1}S_{ik}(\eta) = \eta^{-2i}C_{ik}(\eta), \qquad 0 \leqq i \leqq k.$$

We have just established that the right side is less than unity in magnitude. Hence (8) holds for $\theta=\eta$, and therefore for any $\theta \neq 0$.

The limits (9) are obtained by expanding (4) and (7) in powers of θ, integrating term by term, and noting that

$$\int_{-1}^{1} u_{ik}(x)x^{2j}dx = (2i)!\delta_{ij},$$

$$\int_{-1}^{1} v_{ik}(x)x^{2j+1}dx = (2i+1)!\delta_{ij}, \qquad 0 \leqq i,j \leqq k.$$

References

1. W. F. Trench, *On the stability of midpoint smoothing with Legendre polynomials*, Proc. Amer. Math. Soc. **18** (1966), 191–199.

2. G. Polya and G. Szego, *Aufgaben und Lehrsätze aus der Analysis*, Springer, Berlin, 1925.

Drexel Institute of Technology