ONE CAN HEAR WHETHER A DRUM
HAS FINITE AREA

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In [1] Clark has obtained an asymptotic formula involving the eigenvalues of the Laplacian operator \(-\Delta\) (with zero boundary conditions) on a "quasi-bounded" region \(\Omega\) in \(\mathbb{R}^n\). A region is called quasi-bounded if it cannot contain an infinite family of nonintersecting open solid \(n\)-spheres of equal size. The formula (valid under certain additional assumptions) is as follows.

\[
N_\rho(\lambda) \sim \frac{\lambda}{4\pi} \frac{n/2}{(1 + n/2)} \int_{\Omega} \rho(x) dx,
\]

where \(N_\rho(\lambda) = \sum_{\lambda_j \leq \lambda} \int_{\Omega} \rho(x) (\phi_j(x))^2 dx\), in which \(\rho(x)\) is an arbitrary nonnegative function in \(L_1(\Omega)\), and \(\{\lambda_j\}, \{\phi_j\}\) are the eigenvalues and eigenfunctions, respectively, of the Laplacian in \(\Omega\). (This formula has also been derived, under other conditions than in [1], by Hewgill [2].)

The recent paper by Kac [3] prompts the question: is there some distinction asymptotically between the eigenvalues for a region of finite volume (e.g. a bounded region) and those of a quasi-bounded region of infinite volume?

**Theorem 1.** Let \(\Omega\) be a quasi-bounded region in \(\mathbb{R}^n\) for which the formula (1) holds. Assume that \(\Omega\) has infinite \(n\)-dimensional volume. Then the function \(N(\lambda) = \sum_{\lambda_j \leq \lambda} 1\) satisfies

\[
\lim_{\lambda \to +\infty} \lambda^{-n/2} N(\lambda) = +\infty.
\]

This result answers the question prompted by Kac; for the indicated limit is finite (it equals \(\{(4\pi)^{n/2} \cdot \Gamma(1 + n/2)\}^{-1}\) times the volume of \(\Omega\)) if \(\Omega\) has finite volume.

For the proof, we can take \(\rho(x)\) as the characteristic function of some subset \(\Omega_0 \subset \Omega\), of finite but arbitrarily large volume \(V\). Then

\[
N(\lambda) \geq N_\rho(\lambda) \sim \frac{\lambda}{4\pi} \frac{n/2}{(1 + n/2)} \cdot V,
\]

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so that for sufficiently large $\lambda$, 
$$
\lambda^{-n/2}N(\lambda) > V\left\{(4\pi)^{n/2} \cdot 2\Gamma(1 + n/2)\right\}^{-1}.
$$

Since $V$ is arbitrary, the proof is complete. An alternate but similar proof can be derived from the monotoneity theorem: $\Omega_0 \subset \Omega$ implies $\lambda_j(\Omega) \leq \lambda_j(\Omega_0)$ for all $j$.

Theorem 1 gives a lower bound for the rate of growth of $N(\lambda)$. Using a result of Hewgill, we can also obtain an upper bound. We describe Hewgill's result [2, Theorem 4.3] for the case of two dimensions. Let $\Omega$ be a quasi-bounded region, bounded by three contours: $\gamma_1$, the positive $X$-axis; $\gamma_2$, the curve $y = \phi(x)$, $x > 0$, with $\phi(x) > 0$ and $\phi(x) \to 0$ as $x \to + \infty$; and $\gamma_3$, a bounded contour joining $\gamma_1$ and $\gamma_2$. Some mild additional assumptions are made concerning smoothness of the function $\phi(x)$. Then: if $\phi^k \in L_1(0, \infty)$ for some positive integer $k$, the eigenvalues $\lbrace \lambda_j \rbrace$ of the Laplacian in $\Omega$ satisfy

$$
\sum \lambda_j^{-2k} < \infty.
$$

**Theorem 2.** Let $\Omega$ be a quasi-bounded region, satisfying the hypotheses of [2, Theorem 4.3]. Assume that $\phi^k \in L_1(0, \infty)$ for some integer $k \geq 1$. Then $\lambda^{-2k}N(\lambda)$ is bounded.

**Proof.** Since the sequence $\lbrace \lambda_j^{-2k} \rbrace$ is nonincreasing and $\sum \lambda_j^{-2k} < \infty$, we have $\lambda_j^{-2k} = O(j^{-1})$. Hence $\lambda_j \geq M \cdot j^{1/2k}$ for some $M > 0$, and therefore

$$
N(\lambda) = \sum_{\lambda_j \geq M} 1 \leq \sum_{j \leq (M^{-1}\lambda)^{2k}} 1 = [(M^{-1}\lambda)^{2k}];
$$

this shows that $\lambda^{-2k}N(\lambda) \leq \text{const}$, as asserted.

If we define

$$
g(\Omega) = \inf \lbrace \nu : \lambda^{-\nu}N(\lambda) \text{ is bounded in } \lambda \rbrace,
$$

our results show that (for $n = 2$) $1 \leq g(\Omega) \leq 2k$. It would be interesting to improve this estimate.

**References**

