ONE CAN HEAR WHETHER A DRUM HAS FINITE AREA

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In [1] Clark has obtained an asymptotic formula involving the eigenvalues of the Laplacian operator $-\Delta$ (with zero boundary conditions) on a "quasi-bounded" region Ω in R^n. A region is called quasi-bounded if it cannot contain an infinite family of nonintersecting open solid n-spheres of equal size. The formula (valid under certain additional assumptions) is as follows.

$$N_\rho(\lambda) \sim (\lambda/4\pi)^{n/2}(1/\Gamma(1+n/2))\int_\Omega \rho(x)dx, \tag{1}$$

where $N_\rho(\lambda)=\sum_{\lambda_j\leq\lambda}\int_\Omega\rho(x)(\phi_j(x))^2dx$, in which $\rho(x)$ is an arbitrary nonnegative function in $L_1(\Omega)$, and $\{\lambda_j\}$, $\{\phi_j\}$ are the eigenvalues and eigenfunctions, respectively, of the Laplacian in Ω. (This formula has also been derived, under other conditions than in [1], by Hewgill [2].)

The recent paper by Kac [3] prompts the question: is there some distinction asymptotically between the eigenvalues for a region of finite volume (e.g. a bounded region) and those of a quasi-bounded region of infinite volume?

THEOREM 1. *Let Ω be a quasi-bounded region in R^n for which the formula* (1) *holds. Assume that Ω has infinite n-dimensional volume. Then the function $N(\lambda)=\sum_{\lambda_j\leq\lambda}1$ satisfies*

$$\lim_{\lambda\to+\infty}\lambda^{-n/2}N(\lambda)=+\infty.$$

This result answers the question prompted by Kac; for the indicated limit is finite (it equals $\{(4\pi)^{n/2}\cdot\Gamma(1+n/2)\}^{-1}$ times the volume of Ω) if Ω has finite volume.

For the proof, we can take $\rho(x)$ as the characteristic function of some subset $\Omega_0\subset\Omega$, of finite but arbitrarily large volume V. Then

$$N(\lambda)\geqq N_\rho(\lambda)\sim(\lambda/4\pi)^{n/2}\,(1/\Gamma(1+n/2))\cdot V,$$

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so that for sufficiently large λ,

$$\lambda^{-n/2}N(\lambda) > V\{(4\pi)^{n/2}\cdot 2\Gamma(1+n/2)\}^{-1}.$$

Since V is arbitrary, the proof is complete. An alternate but similar proof can be derived from the monotoneity theorem: $\Omega_0 \subset \Omega$ implies $\lambda_j(\Omega) \leqq \lambda_j(\Omega_0)$ for all j.

Theorem 1 gives a lower bound for the rate of growth of $N(\lambda)$. Using a result of Hewgill, we can also obtain an upper bound. We describe Hewgill's result [2, Theorem 4.3] for the case of two dimensions. Let Ω be a quasi-bounded region, bounded by three contours: γ_1, the positive X-axis; γ_2, the curve $y=\phi(x)$, $x>0$, with $\phi(x)>0$ and $\phi(x)\to 0$ as $x\to+\infty$; and γ_3, a bounded contour joining γ_1 and γ_2. Some mild additional assumptions are made concerning smoothness of the function $\phi(x)$. Then: *if $\phi^k \in L_1(0, \infty)$ for some positive integer k, the eigenvalues $\{\lambda_j\}$ of the Laplacian in Ω satisfy*

$$\sum \lambda_j^{-2k} < \infty.$$

THEOREM 2. *Let Ω be a quasi-bounded region, satisfying the hypotheses of [2, Theorem 4.3]. Assume that $\phi^k \in L_1(0, \infty)$ for some integer $k \geqq 1$. Then $\lambda^{-2k}N(\lambda)$ is bounded.*

PROOF. Since the sequence $\{\lambda_j^{-2k}\}$ is nonincreasing and $\sum \lambda_j^{-2k} < \infty$, we have $\lambda_j^{-2k} = O(j^{-1})$. Hence $\lambda_j \geqq M\cdot j^{1/2k}$ for some $M>0$, and therefore

$$N(\lambda) = \sum_{\lambda_j \leqq \lambda} 1 \leqq \sum_{j \leqq (M^{-1}\lambda)^{2k}} 1 = [(M^{-1}\lambda)^{2k}];$$

this shows that $\lambda^{-2k}N(\lambda) \leqq \text{const}$, as asserted.

If we define

$$g(\Omega) = \inf\{\nu : \lambda^{-\nu}N(\lambda) \text{ is bounded in } \lambda\},$$

our results show that (for $n=2$) $1 \leqq g(\Omega) \leqq 2k$. It would be interesting to improve this estimate.

REFERENCES

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