INVARIANT MEANS AND FACTOR-SEMIGROUPS[1]

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1. Introduction. A left invariant mean on a discrete semigroup S is a positive, normalized, left-translation-invariant linear functional on the Banach space $m(S)$ of all bounded, real-valued functions on S under the supremum norm. It is well known [**1**] that if S possesses a left invariant mean, then so does every homomorphic image of S; here we consider the problem of reversing this process.

The problem of lifting invariant means on quotients of a semigroup S to S itself was introduced by Dixmier [**2**]. Let A be a *distinguished subsemigroup* of a semigroup S. (Define a relation (r) on S as follows: $s(r')t$ if and only if $sA\cap tA\neq\emptyset$, and $s(r)t$ if and only if there exist $u_1, u_2, \cdots, u_n$ in S such that $s(r')u_1, u_1(r')u_2, \cdots, u_n(r')t$. A similar relation can be defined on the right, and A is called *distinguished* if the two sets of classes coincide.) Dixmier showed that if A and S/A both possess left invariant means, then so does S.

In [**3**], E. Granirer observed that if S is a semigroup in which each pair of right ideals has a nonvoid intersection, then the relation (r) on S, defined by $s(r)t$ if and only if there exists $u\in S$ such that $su=tu$, is a two-sidedly stable equivalence relation on S and hence $S/(r)$ forms a semigroup under the natural multiplication [**5**]. The purpose of this paper is to prove that if $S/(r)$ has a left invariant mean, then so does S.

2. Notation and preliminaries. The only topology we consider in $m(S)^*$ is the w^*-topology. Let q be the natural embedding of S into $m(S)^*$, i.e., $qs(x)=x(s)$ for all $s\in S$ and $x\in m(S)$. Then the closure of qS in $m(S)^*$ coincides with the Stone-Čech compactification, βS, of the discrete space S. If $A\subseteq S$, denote the closure of qA in βS by $\mathrm{Cl}(A)$.

Following Day [**1**], we define the Arens product of two means as follows. For $s\in S$ let l_s be the left-translation operator in $m(S)$; i.e., $l_s x(t)=x(st)$ for $t\in S$. The Arens product, $\mu\odot\nu$, of two means μ, ν is defined on $m(S)$ by $\mu\odot\nu(x)=\mu(F_\nu x)$, where $F_\nu x(s)=\nu(l_s x)$ for $s\in S$. The Arens product renders βS and the set of all means on $m(S)$ semigroups, each containing qS as a subsemigroup. A mean μ is left invariant if and only if $qs\odot\mu=\mu$ for all $s\in S$.

3. The main theorem. The purpose of this section is to prove our principal result:

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THEOREM. *Let S be a semigroup in which each pair of right ideals has a nonvoid intersection. Define a relation (r) in S by $s(r)t$ if and only if there exists $u \in S$ such that $su = tu$; then $S/(r)$ is a semigroup. If $S/(r)$ has a left invariant mean, then so does S.*

Through the proof S and (r) will be as described in the theorem.

LEMMA 1 (GRANIRER [3]). *(r) is an equivalence relation and is stable on both sides, i.e., $s(r)t$ implies $su(r)tu$ and $us(r)ut$ for s, t, u in S; hence $S/(r)$ is a semigroup.*

The proof used by Granirer will be illustrated in Lemma 2.

LEMMA 2. *Let F be a finite subset of S such that $s(r)t$ for all $s, t \in F$. Then there exists $u \in S$ such that Fu consists of one element.*

PROOF. Let $s(r)t$, $t(r)u$; take $v, w \in S$ such that $sv = tv$, $tw = uw$. Take $v', w' \in S$ such that $vv' = ww'$; then $svv' = tvv' = tww' = uww' = uvv'$. The induction process has now been demonstrated.

First, T. Mitchell [6], then E. Granirer [4] showed that if $S/(r)$ consists of a single class, then S has a left invariant mean which, in fact, lies in βS. In Lemma 3 we obtain points in βS which, although not translation-invariant, are class-independent for each class.

LEMMA 3. *There exists a point ω in βS such that $a(r)b$ in S implies $qa \odot \omega = qb \odot \omega$.*

PROOF. Let $\{A_1, \cdots, A_k\}$ be a finite subset of $S/(r)$. Let $\mathfrak{F}_i$ be the family of all finite subsets of A_i (as a set) for $i = 1, \cdots, k$, each directed upward by inclusion, let $\mathfrak{F}$ be the product directed set, and define a net over $\mathfrak{F}$ as follows. If $F \in \mathfrak{F}$, then $F = (F_1, \cdots, F_k)$ with $F_i \in \mathfrak{F}_i$ for each i. Then there exist points $u_i, v_i \in S$ such that $F_i u_i = \{v_i\}$ for $i = 1, \cdots, k$. Since each finite collection of right ideals in S has a nonvoid intersection, there exist points $w_i \in S$ such that $u_1 w_1 = \cdots = u_k w_k$; call this common product s_F. Then for each i, $F_i s_F = F_i u_i w_i = \{v_i w_i\}$; let $v_i w_i = t_F^i$. Let ω be a cluster point of the net $\{qs_F, F \in \mathfrak{F}\}$, and let $\{qx_n, n \in N\}$ be a subset converging to ω. We shall show that for each i the net $\{qy_n^i, n \in N\}$ (here $x_n = s_{F_n}$, where $n \to F_n$ is the subnet function, and y_n^i is the corresponding point $t_{F_n}^i$) converges, say to Ψ_i, in βS, and that $qa_i \odot \omega = \Psi_i$ for each $a_i \in A_i$. Fix any $a_0^i \in A_i$ and let $\Psi_i = qa_0^i \odot \omega$. Let $F_0 = (\{a_0^1\}, \cdots, \{a_0^k\})$; by definition of subnet, there exists $n_0 \in N$ such that $F_n \geqq F_0$ when $n \geqq n_0$ (i.e., $a_0^i \in F_n^i$ for each i, where $F_n = (F_n^1, \cdots, F_n^k)$). Thus $a_0^i x_n = y_n^i$ or $qa_0^i \odot qx_n = y_n^i$. Since $\lim_n (qa_0^i \odot qx_n) = qa_0^i \odot \omega = \Psi_i$ by continuity of the multiplication operator, we have $\lim_n y_n^i = \Psi_i$ for $i = 1, \cdots, k$. For arbitrary $a_i \in A_i$, repeat the argument using subnets to obtain $qa_i \odot qx_n = qa_0^i \odot qx_n$, $i = 1, \cdots, k$, for sufficiently large n. Then $\lim_n (qa_i \odot qx_n) = \Psi_i$; hence $qa_i \odot \omega = \Psi_i$ for each i. If we now let $T(A)$ for any $A \in S/(r)$ be the set of all $\omega \in \beta S$ which are class-independent for A, then each $T(A)$ is closed, and the above argument shows that the family $\{T(A): A \in S/(r)\}$ has the finite intersection property. The lemma now follows from compactness of βS.

Proof of main theorem. Let $\omega \in \beta S$ be class-independent for each $A \in S/(r)$ by Lemma 3, and for each such A let $\omega_A = qa \odot \omega$, where a is any representative of A. Define a mapping T on $m(S)$ to $m(S/(r))$ by $Tx(B) = \omega_B(x)$. Then T is positive, linear and preserves the constant functions. Let Ψ be a left invariant mean on $m(S/(r))$ and define $\mu = T^*\Psi$; then μ is a mean on $m(S)$. If $s \in S$, then $[T(l_s x)](B) = \omega_B(l_s x) = qsb \odot \omega(x) = \omega_{AB}(x)$, where A is the class of s. Thus $[T(l_s x)](B) = Tx(AB) = \lambda_A Tx(B)$, where λ is the left-translation operator in $m(S/(r))$. Now clearly $\mu(l_s x) = \mu(x)$.

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