CONFORMAL EQUIVALENCE OF COUNTABLE DENSE SETS

W. D. MAURER

In [1, p. 297, problem 24], Erdös asks:

"Does there exist an entire function f, not of the form $f(z)=a_0+a_1z$, such that the number $f(x)$ is rational or irrational according as x is rational or irrational? More generally, if A and B are two denumerable, dense sets, does there exist an entire function which maps A onto B?"

The following theorem settles the second part of this question as it is stated.

THEOREM. *Let A and B be two countable dense sets in the complex plane. Then there exists an entire function taking A onto B.*

PROOF. Let a and b be enumerations of A and B, i.e., $A=\{a_1, a_2, \cdots\}$, $B=\{b_1, b_2, \cdots\}$. We construct two new enumerations c and d of A and B respectively, together with a sequence f_i of polynomials such that $f(c_i)=d_i$ for each i, where $f=\sum_{j=1}^{\infty} f_j$.

The construction is as follows. Let $c_1=a_1$, $d_1=b_1$, $f_1=d_1$ (the constant function). At the $(2n-1)$st stage, suppose that $c_1, \cdots, c_{2n-1}$; $d_1, \cdots, d_{2n-1}$; and $f_1, \cdots, f_{2n-1}$ have been chosen, such that $g_{2n-1}(c_i)=d_i$, $1\leqq i\leqq 2n-1$, where $g_{2n-1}=\sum_{j=1}^{2n-1} f_j$. Let $c_{2n}=a_j$, where j is the smallest index such that $a_j\neq c_i$, $1\leqq i\leqq 2n-1$, and set $y_{2n}=g_{2n-1}(c_{2n})$. Let the function $h_{2n-1}=(z-c_1)(z-c_2)\cdot\ \cdots\ \cdot(z-c_{2n-1})$, and consider the functions $g_{2n-1}+k_{2n-1}h_{2n-1}$, for

$$|k_{2n-1}| < \frac{1}{(2n-1)!u_1u_2\cdot\ \cdots\ \cdot u_{2n-1}} = m_{2n-1}$$

where $u_i=\max(1, |c_i|)$, $1\leqq i\leqq 2n-1$. These functions map c_{2n} into $\{y_{2n}+k_{2n-1}h_{2n-1}(c_{2n}): |k_{2n-1}|<m_{2n-1}\}$, which is a neighborhood of y_{2n} because $h_{2n-1}(c_{2n})\neq 0$, and consequently contains an element of the dense set $B-\{d_1, \cdots, d_{2n-1}\}$, which we denote by d_{2n}. For the corresponding value of k_{2n-1}, we set $f_{2n}=k_{2n-1}h_{2n-1}$, $g_{2n}=g_{2n-1}+f_{2n}$, which clearly implies $g_{2n}(c_i)=d_i$, $1\leqq i\leqq 2n$, and $g_{2n}=\sum_{j=1}^{2n} f_j$. This brings us to the $2n$th stage. Let $d_{2n+1}=b_j$, where j is the smallest index such that $b_j\neq d_i$, $1\leqq i\leqq 2n$, and set x_{2n+1} such that $g_{2n}(x_{2n+1})=d_{2n+1}$; this is always possible since g_{2n} is a polynomial. Let the function $h_{2n}=(z-c_1)(z-c_2)\cdot\ \cdots\ \cdot(z-c_n)$, and consider the functions $g_{2n}+k_{2n}h_{2n}$, for

Received by the editors January 25, 1966.

$$|k_{2n}| < \frac{1}{(2n)!u_1u_2 \cdot \cdots \cdot u_{2n}} = m_{2n}$$

where $u_i = \max(1, |c_i|)$, $1 \leqq i \leqq 2n$. These functions map all elements of some neighborhood of x_{2n+1} into d_{2n+1}, and hence there exists a particular value of k_{2n} for which $g_{2n+1}(c_{2n+1}) = d_{2n+1}$, where $f_{2n+1} = k_{2n}h_{2n}$, $g_{2n+1} = g_{2n} + f_{2n+1}$, and c_{2n+1} is a member of the dense set $A - \{c_1, \cdots, c_{2n}\}$.

The functions $|f_j|$ are majorized by

$$\frac{(z-c_1)(z-c_2) \cdots (z-c_j)}{j!u_1u_2 \cdot \cdots \cdot u_j} = \frac{1}{j!}\left(\frac{z-c_1}{u_1}\right)\left(\frac{z-c_2}{u_2}\right) \cdots \left(\frac{z-c_j}{u_j}\right).$$

For each i, $1 \leqq i \leqq j$, if $|c_i| \leqq 1$, then $u_i = 1$ and $|(z-c_i)/u_i| = |z-c_i| \leqq |z| + 1$, while if $|c_i| > 1$, then $u_i = |c_i|$ and $|(z-c_i)/u_i| \leqq |z/|c_i|| + |c_i/|c_i|| = |z/|c_i|| + 1 < |z| + 1$. Thus the functions $|f_j|$ are also majorized by $(|z|+1)^j/j!$, and therefore $f = \sum_{j=1}^{\infty} f_j$ is an entire function with $f(c_i) = d_i$ for all i. By virtue of the "back-and-forth" induction, the maps c and d are enumerations, i.e., $A = \{c_1, c_2, \cdots\}$ and $B = \{d_1, d_2, \cdots\}$, since in fact $\{c_1, \cdots, c_{2n}\} \supseteq \{a_1, \cdots, a_n\}$ and $\{d_1, \cdots, d_{2n}\} \supseteq \{b_1, \cdots, b_n\}$ for each n. Therefore f takes A onto B.

In particular, this gives a negative answer to the question posed by F. Gross in [2]. A more general question remains open, which in one sense is a closer generalization of the first question asked by Erdös:

Let A and B be two denumerable, dense subsets of the complex plane. Does there exist an entire function which maps A onto B and $\overline{A}$ onto $\overline{B}$?

According to the proof given above, we can say only that there exists a function whose restriction to A gives a one-to-one map from A to B. The author is grateful to the referee for his helpful comments on this paper.

References

1. Paul Erdös, *Some unsolved problems*, Michigan Math. J. **4** (1957), 291–300.

2. Fred Gross, *Function theory*, Research Problem 19, Bull. Amer. Math. Soc. **71** (1965), 853.

University of California, Berkeley