CONFORMAL EQUIVALENCE OF COUNTABLE DENSE SETS

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In [1, p. 297, problem 24], Erdös asks:

"Does there exist an entire function $f$, not of the form $f(z) = a_0 + a_1 z$, such that the number $f(x)$ is rational or irrational according as $x$ is rational or irrational? More generally, if $A$ and $B$ are two denumerable, dense sets, does there exist an entire function which maps $A$ onto $B"?"

The following theorem settles the second part of this question as it is stated.

**Theorem.** Let $A$ and $B$ be two countable dense sets in the complex plane. Then there exists an entire function taking $A$ onto $B$.

**Proof.** Let $a$ and $b$ be enumerations of $A$ and $B$, i.e., $A = \{a_1, a_2, \ldots\}$, $B = \{b_1, b_2, \ldots\}$. We construct two new enumerations $c$ and $d$ of $A$ and $B$ respectively, together with a sequence $f_i$ of polynomials such that $f(c_i) = d_i$ for each $i$, where $f = \sum_{j=1}^{\infty} f_j$.

The construction is as follows. Let $c_1 = a_1$, $d_1 = b_1$, $f_1 = d_1$ (the constant function). At the $(2n-1)$st stage, suppose that $c_1, \ldots, c_{2n-1}$; $d_1, \ldots, d_{2n-1}$; and $f_1, \ldots, f_{2n-1}$ have been chosen, such that $g_{2n-1}(c_i) = d_i$, $1 \leq i \leq 2n-1$, where $g_{2n-1} = \sum_{j=1}^{2n-1} f_j$. Let $c_{2n} = a_j$, where $j$ is the smallest index such that $a_j \neq c_i$, $1 \leq i \leq 2n-1$, and set $y_{2n} = g_{2n-1}(c_{2n})$. Let the function $h_{2n-1} = (z - c_1)(z - c_2)\cdots(z - c_{2n-1})$, and consider the functions $g_{2n-1} + h_{2n-1}$, for

$$|k_{2n-1}| < \frac{1}{(2n-1)!u_1u_2\cdots u_{2n-1}} = m_{2n-1}$$

where $u_i = \max(1, |c_i|)$, $1 \leq i \leq 2n-1$. These functions map $c_{2n}$ into $\{y_{2n} + k_{2n-1}h_{2n-1}(c_{2n}) : |k_{2n-1}| < m_{2n-1}\}$, which is a neighborhood of $y_{2n}$ because $h_{2n-1}(c_{2n}) \neq 0$, and consequently contains an element of the dense set $B - \{d_1, \ldots, d_{2n-1}\}$, which we denote by $d_{2n}$. For the corresponding value of $k_{2n-1}$, we set $f_{2n} = k_{2n-1}h_{2n-1}$, $g_{2n} = g_{2n-1} + f_{2n}$, which clearly implies $g_{2n}(c_i) = d_i$, $1 \leq i \leq 2n$, and $g_{2n} = \sum_{j=1}^{2n} f_j$. This brings us to the 2nth stage. Let $d_{2n+1} = b_j$, where $j$ is the smallest index such that $b_j \neq d_i$, $1 \leq i \leq 2n$, and set $x_{2n+1}$ such that $g_{2n}(x_{2n+1}) = d_{2n+1}$; this is always possible since $g_{2n}$ is a polynomial. Let the function $h_{2n} = (z - c_1)(z - c_2)\cdots(z - c_n)$, and consider the functions $g_{2n} + k_{2n}h_{2n}$, for

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where \( u_i = \max(1, |c_i|), 1 \leq i \leq 2n \). These functions map all elements of some neighborhood of \( x_{2n+1} \) into \( d_{2n+1} \), and hence there exists a particular value of \( k_{2n} \) for which \( g_{2n+1}(c_{2n+1}) = d_{2n+1} \), where \( f_{2n+1} = k_{2n}h_{2n}, g_{2n+1} = g_{2n} + f_{2n+1} \), and \( c_{2n+1} \) is a member of the dense set \( A = \{ c_1, \ldots, c_{2n} \} \).

The functions \(|f_j|\) are majorized by

\[
\left| \frac{(z - c_1)(z - c_2) \cdots (z - c_j)}{j!u_1u_2 \cdots u_j} \right| = \frac{1}{j!} \left( \frac{z - c_1}{u_1} \right) \left( \frac{z - c_2}{u_2} \right) \cdots \left( \frac{z - c_j}{u_j} \right).
\]

For each \( i, 1 \leq i \leq j \), if \( |c_i| \leq 1 \), then \( u_i = 1 \) and \( \left| (z - c_i)/u_i \right| = |z - c_i| \leq |z| + 1 \), while if \( |c_i| > 1 \), then \( u_i = |c_i| \) and \( \left| (z - c_i)/u_i \right| \leq |z|/|c_i| + |c_i|/|c_i| = |z|/|c_i| + 1 \). Thus the functions \(|f_j|\) are also majorized by \( (|z| + 1)^j/j! \), and therefore \( f = \sum_{j=1}^{\infty} f_j \) is an entire function with \( f(c_i) = d_i \) for all \( i \). By virtue of the “back-and-forth” induction, the maps \( c \) and \( d \) are enumerations, i.e., \( A = \{ c_1, c_2, \ldots \} \) and \( B = \{ d_1, d_2, \ldots \} \), since in fact \( \{ c_1, \ldots, c_{2n} \} \supseteq \{ a_1, \ldots, a_n \} \) and \( \{ d_1, \ldots, d_{2n} \} \supseteq \{ b_1, \ldots, b_n \} \) for each \( n \). Therefore \( f \) takes \( A \) onto \( B \).

In particular, this gives a negative answer to the question posed by F. Gross in [2]. A more general question remains open, which in one sense is a closer generalization of the first question asked by Erdös:

Let \( A \) and \( B \) be two denumerable, dense subsets of the complex plane. Does there exist an entire function which maps \( A \) onto \( B \) and \( A \) onto \( B \)?

According to the proof given above, we can say only that there exists a function whose restriction to \( A \) gives a one-to-one map from \( A \) to \( B \). The author is grateful to the referee for his helpful comments on this paper.

References


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