A MINIMAL COMPACTIFICATION FOR EXTENDING CONTINUOUS FUNCTIONS

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Let W be a noncompact, locally compact Hausdorff space, and let Q be a nonvoid set of continuous functions defined on W with each $f \in Q$ having its range in some compact Hausdorff space X_f. Following the work of Constantinescu and Cornea ([**1**, pp. 96–97], where W is a Riemann surface and Q, a class of extended real-valued functions) one adjoins the continuous real-valued functions with compact support, C_0, to Q and associates W with its image in the product space $\prod_{f \in Q \cup C_0} X_f$ under the evaluation map e. (For each $x \in W$, $e(x)(f) = f(x)$.) The closure of $e(W)$ in $\prod_{f \in Q \cup C_0} X_f$ is a compact Hausdorff space $\overline{W}$ containing W (i.e. $e(W)$) as a dense subset, and the functions in Q have continuous extensions to $\overline{W}$ which separate the points of $\overline{W} - W$. Any compactification of W with these properties is homeomorphic to this one and is called a Q-compactification of W.

Below, we construct a Q-compactification of W which uses only the product of the spaces X_f for f in Q. Thus we avoid using the axiom of choice if Q is a countable collection of real- or complex-valued functions (see [**3**]), and for many examples we obtain a compactification which is easier to visualize than the one given above.

THEOREM. *Let Y be the product space $\prod_{f \in Q} X_f$ and e the evaluation map sending W into Y. Set $\Delta = \bigcap \{ [e(W-K)]^- : K \text{ compact}, K \subset W \}$, and let $\tilde{W} = W \cup \Delta$. Given a point x in Δ, an open set N in the standard base for the topology of Y with $x \in N$, and a compact set $K \subset W$, we set $N(x, K) = [N \cap \Delta] \cup [e^{-1}(N) - K]$. If $\mathfrak{I}$ is the topology on $\tilde{W}$ generated by the base consisting of all open sets in W and all the sets $N(x, K)$, then $(\tilde{W}, \mathfrak{I})$ is a Q-compactification of W.*

The proof of the theorem is not hard; the compactness of $\tilde{W}$ follows from the fact that a net which is eventually in the complement of every compact subset of W has a cluster point in Δ. (See [**2**, p. 136].) Note that if Q is the restriction of the continuous real-valued functions with period 1 to the interval $W = \{x : 0 < x \leqq 1\}$, then $e(W)$, which is compact, is not homeomorphic to W even though Q separates the points of W. Moreover, Δ consists of the single point $e(1)$, which is distinct from 1 in $\tilde{W}$. If $W = \{x : 0 < x \leqq 1\}$ and

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$Q = \{\sin(\pi/x)\}$, then $\Delta = \{y: -1 \leqq y \leqq 1\}$ and a typical basic neighborhood of a point $y_0 \in \Delta$ is given by positive constants δ and ϵ and has the form

$$\{y \in \Delta: |y - y_0| < \epsilon\} \cup \{x \in W: x < \delta \text{ and } |\sin(\pi/x) - y_0| < \epsilon\}.$$

References

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3. P. A. Loeb, *A new proof of the Tychonoff theorem*, Amer. Math. Monthly **72** (1965), 711–717.

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