COVERING THE TRACK OF AN ISOTOPY

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We work in the Polyhedral Category as defined in [3], which consists of polyhedra and polymaps. A polyhedron is a space with a maximal nonempty family of p.l. related triangulations and a polymap is p.l. w.r.t. these triangulations.

An *ambient isotopy* of a polyhedron X is a level-preserving homeo,

$H: X\times I\to X\times I$ with $H|X\times 1=1$,

$H_t: X\to X$ for $t\in I$ is the homeo induced by $H|X\times t$.

An *isotopy of Y in X* is a level-preserving embedding,

$F: Y\times I\to X\times I$,

$F_t: Y\to X$ for $t\in I$ is the embedding induced by $F|Y\times t$.

We say that H *covers* F if $H_tF_0=F_t$ for all $t\in I$, and we say that H *covers the track of* F if $H_tF_0Y=F_tY$ for all $t\in I$.

Given an isotopy F of Y in X, it is not true that there is always an ambient isotopy of X which covers F or even just the track of F. For example, if we restrict attention to manifolds and proper embeddings, then classical knots of S^1 in S^3 are isotopic embeddings which are not ambient isotopic.

Ambient isotopy is rather more useful than isotopy, but isotopies are usually easier to construct. The following problem is therefore of interest.

Problem A. Given an isotopy F of Y in X, under what conditions can we cover F by an ambient isotopy of X?

For some purposes it is enough to cover the track of F, e.g. when considering knots as *subspaces* rather than *embeddings*, i.e. working *set*-wise rather than *point*-wise; so we have the weaker problem.

Problem B. Under what conditions can we cover the track of an isotopy of Y in X?

In [1] Zeeman and Hudson give a solution to A for proper embeddings of manifolds. Their condition for coverability is local-unknotting of $F|Y\times J$ for any subinterval $J\subset I$, which is always true in codimension $\geqq 3$, see [1] for the precise definition.

In this paper we will be concerned only with Problem B and our main result is that any *locally collarable* isotopy (definition below) can be track-covered.

Local collarability is a necessary condition for track-covering and, in the case of proper embeddings of manifolds, is strictly weaker than local unknotting.

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The proof of our result rests heavily on techniques of Zeeman [2] and Hudson and Zeeman [1]; however it improves on known answers to Problem B, since it works for any compact polyhedra (not merely for manifolds).

1. **Collaring and local collarability.** A *pair* $\{X\}$ consists of a polyhedron X with subpolyhedron $X_1 \subset X$. $C(\{X\})$ denotes the cone (pair) on $\{X\}$.

Collaring. Let $\{Y\} \subset \{X\}$ be a subpair (i.e. $Y \subset X$ and $Y_1 \subset X_1$), $\{Y\}$ is said to be *collared* or *collarable* in $\{X\}$ if $\exists$ a homeo $h: \{M\} \to \{X\}$ with $h|(\{Y\} \times 0) =$ canon. proj. $\{Y\} \times 0 \to \{Y\}$, where $\{M\}$ denotes the mapping cylinder of $\{Y\} \subset \{X\}$ i.e. $\{M\} = \{Y\} \times I \cup_{\{Y\} \times 1 = \{Y\}} \{X\}$.

Local Collaring. Let $\{Y\} \subset \{X\}$ and let P be a point in Y, $\{Y\}$ is said to be *locally collared in* $\{X\}$ *at* P if $\mathrm{lk}(P, \{X\}) \cong C(\mathrm{lk}(P, \{Y\}))$, as pairs, in any (and $\therefore$ every) triangulation with P a vertex, where the homeo restricts to 1 on $\mathrm{lk}(P, \{Y\})$.

$\{Y\}$ is said to be *locally collarable* in $\{X\}$ if it is locally collared at each point in Y. We have

Collaring Theorem. *Let $\{Y\} \subset \{X\}$ be both compact (i.e. X, X_1, Y, Y_1 all compact), then $\{Y\}$ is collarable in $\{X\}$ $\Leftrightarrow$ it is locally collarable in X.*

Remark. If $\{X\}$ is a properly embedded manifold pair, and $\{Y\}$ is the boundary pair, then local unknotting on the boundary in the sense of [2] $\Rightarrow$ local collarability (straight from the definitions).

However in this case local collarability is a strictly weaker condition, e.g. take $S^2 \subset S^4$ locally knotted somewhere, say at a point Q, and define $M = S^4 \times I$ and $M_1 = S^2 \times I \subset M$, then the pair $\{M\} = M_1 \subset M$ is collarable on its boundary and hence locally collarable but it is locally knotted on the boundary at $Q \times 0$ and $Q \times 1$.

Our Collaring Theorem therefore improves directly on Zeeman's [2] (for locally unknotted properly embedded manifold pairs), the proof however is almost identical.

Proof of the Collaring Theorem. $\Rightarrow$ is trivial, so we just prove $\Leftarrow$. Triangulate $\{X\}$ and $\{Y\}$ by complexes $\{K\}$, $\{L\}$, i.e.

$$\begin{array}{ccc} L & \subset & K \\ \cup & & \cup \\ L_1 & \subset & K_1, \end{array}$$

all simplicial, and let $\{K'\}$, $\{L'\}$ denote barycentric first deriveds.

Order the simplexes of L in order of increasing dimension as $A_1, A_2, \cdots, A_t$. Denote the barycentre of A_i by P_i for each i, and define the *dual* (complex) of A_i in L' for each i as follows (denoted by A_i^*),

$$A_i^* = \bigcap_{P \text{ a vertex of } A_i} \bar{s}t(P, L').$$

A_i^* is a cone vertex P_i for each i, and we write the base $A_i^\sim$, so $A_i^* = P_i \cdot A_i^\sim$.

Write $\{A_i\}$ etc. for the corresponding subpair of $\{Y\}$ to A_i etc., i.e. $\{A_i\} = A_i \cap Y_1 \subset A_i$. Let $\{M\}$ be the mapping cylinder of $\{Y\} \subset \{X\}$ (defined above) and using the linear structure of the prisms $A_j \times I$ define, for each i, the join $A_i^+ = A_i \times 0 \cdot A_i^* \times 1$, and define inductively

$$M^0 = X, \qquad Y^0 = Y, \qquad M^i = M^{i-1} \cup A_i^+,$$

$$Y^i = (Y^{i-1} - \partial A_i \times 0 \cdot A_i^* \times 1) \cup A_i \times 0 \cdot A_i^\sim \times 1;$$

loosely think of Y^i as obtained from Y^{i-1} by moving $P_i \times 1$ up to $P_i \times 0$.

Note that $M^t = M$ and $Y^t = Y \times 0$; denote by $\{A_i^+\}\{M^i\}$, $\{Y^i\}$ the corresponding subpairs of $\{M\}$.

Now there is a natural homeo π_i: $\{Y^i\} \to \{Y\}$ for each i induced by projection $\{Y\} \times I \to \{Y\}$ (think of it as moving the $P_j \times 0$ back down to $P_j \times 1$ in reverse order); note that $\pi_0 = 1$.

We will define homeos h_i: $\{M^i\} \to \{X\}$ for each i such that $h_i | \{Y^i\} = \pi_i$, and this is by induction on i starting with $h_0 = 1$. h_t: $\{M\} \to \{X\}$ will be the required homeo for the definition of collaring since $h_t | (\{Y\} \times 0) = \pi_t | (\{Y\} \times 0) =$ projection.

So suppose h_{i-1} to be defined as above and define h_i. Note that

$$M^i = M^{i-1} \cup A_i^+ = M^{i-1} \cup (P_i \times 0 \cdot \partial A_i \times 0 \cdot A_i^* \times 1);$$

$$h_{i-1}(\partial A_i \times 0 \cdot A_i^* \times 1) = \pi_{i-1}(\partial A_i \times 0 \cdot A_i^* \times 1) = \bar{s}t(P_i, L').$$

Therefore h_{i-1} and conical extension (vertex $P_i \times 0$) gives a homeo

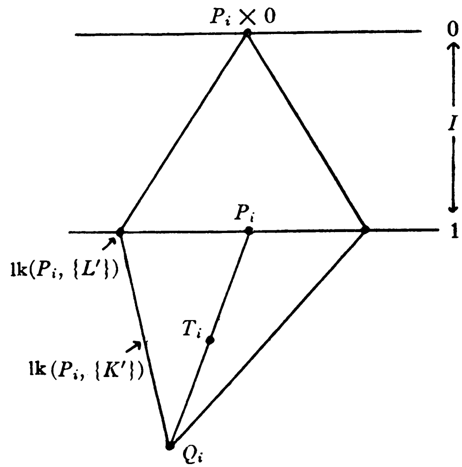
$$\bar{h}_{i-1}: \{M^i\} \to \{X\} \cup P_i \times 0 \cdot \bar{s}t(P_i, \{L'\}).$$

Now

$$\bar{s}t(P_i, \{K'\}) = P_i \cdot \text{lk}(P_i, \{K'\}) = P_i \cdot C(\text{lk}(P_i, \{L'\}))$$

by local collarability. Let the vertex of $C(\text{lk}(P_i, \{L'\}))$ be Q_i;$P_i \cdot C(\text{lk}(P_i, \{L'\}))$ can be thought of as a cone pair vertex at some point T_i on the ray $Q_i P_i$, and this represents $\bar{s}t(P_i, \{K'\})$ as $T_i \cdot \Sigma\ \text{lk}(P_i, \{L'\})$ [Σ denotes suspension].

But $P_i \times 0 \cdot \bar{s}t(P_i, \{L'\}) \cup \bar{s}t(P_i, \{K'\}) \cong P_i \cdot \Sigma\ \text{lk}(P_i, \{L'\})$. So we



can define a homeo β_i: $P_i \times 0 \cdot \bar{s}t(P_i, \{L'\}) \cup \bar{s}t(P_i, \{K'\}) \to \bar{s}t(P_i, \{K'\})$ as follows: map $P_i \times 0 \cdot \mathrm{lk}(P_i, \{L'\})$ conically to $P_i \cdot \mathrm{lk}(P_i, \{L'\})$, map $\mathrm{lk}(P_i, \{K'\})$ by the identity, and extend to the rest conically, vertices P_i and T_i respectively. Extend β_i by the identity on $\{X\} - \bar{s}t(P_i, \{K'\})$, and define $h_i = \beta_i \bar{h}_{i-1}$ and the required condition is just by the construction of β_i.

This completes the Collaring Theorem.

Addenda. This section works just as well for *flags*, i.e. sequences $\{X\} = X_n \subset X_{n-1} \subset \cdots \subset X_1 \subset X$, without alteration.

The Collaring Theorem also works with some of the compactness conditions relaxed (we leave the reader to investigate if interested).

2. **Covering the track of an isotopy.** An isotopy F: $Y \times I \to X \times I$ is said to be *locally collarable* if, for each subinterval $J \subset I$, the pair $F(Y \times \partial J)$, $X \times \partial J$ is locally collarable in $F(Y \times J)$, $X \times J$ (∂J denotes the boundary of J). Our main result is

TRACK COVERING ISOTOPY THEOREM. *Let X, Y be compact and F an isotopy of Y in X.*

$\exists$ an ambient isotopy of X covering the track of F;

$\Leftrightarrow F$ is locally collarable.

REMARK. As in the Collaring Theorem, in the case of proper embeddings of manifolds, local collarability is strictly weaker than local unknotting.

PROOF OF THE THEOREM $\Rightarrow$ an ambient isotopy collars $F(Y \times \partial J)$, $X \times \partial J$ in $F(Y \times J)$, $X \times J$ for each $J \subset I$, and therefore implies local collarability. $\Leftarrow$ comes quickly from two of the techniques of Hudson and Zeeman [1], as follows:

PROPOSITION. *There is a short ambient isotopy of X which covers the track of the beginning of F.*

PROOF. By local collarability and the Collaring Theorem, the pair $F(Y\times 0)$, $X\times 0$ is collarable in $F(Y\times I)$, $X\times I$.

This implies that $\exists$ an embedding, c: $X\times I$, $F_0Y\times I\rightarrow X\times I$, $F(Y\times I)$ which maps $X\times I$ onto a neighborhood of $X\times 0$ in $X\times I$.

Now by [**1**, Lemma 5] we can change c to c^* with c^* level preserving in $[0, \delta]$; the method of proof is to star c on the δ level for all simplexes meeting the δ level, for δ smaller than all meshes in some triangulation of c.

Note therefore that $c^*(X\times[0, \delta])$ is still a neighborhood of $X\times 0$ in $X\times I$ and that $c^*(F_0Y\times I)$ is still $\subset F(Y\times I)$.

$c^*(c_0^{*-1}\times 1)|[0,\delta]$ is therefore a short ambient isotopy of X covering the beginning of the track of F, proving the proposition.

Now by the proposition and the fact that local collarability gives the same condition on either side of each level, we get, for each $t\in I$, a short "ambient isotopy" H^t of X, which covers the track of F in a neighborhood of t.

The theorem therefore follows by the argument in [**1**, proof of Theorem 2], which is a compactness argument. By compactness of I, we get a set of abutting intervals covering I with "short ambient isotopies" on each covering of the track of F; we form H by piecing them together, taking care to make them agree where they meet—see [**1**] for details.

Addenda. (1) We can choose H to keep fixed the complement of a neighborhood of the *track* of F in X $(=\bigcup_t F_tY\subset X)$.

This is because in the Collaring Theorem, as applied in the above proof, we can choose our collar to be standard outside a neighborhood of $F(Y\times 0)$, which gives the short ambient isotopy to be the identity outside a neighborhood of F_0Y.

(2) Some of the compactness conditions can be relaxed in the theorem (e.g. by (1) it is enough for a neighborhood of the track of F to be compact).

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