AN ELEMENTARY PROOF OF INTEGRATION BY PARTS FOR THE PERRON INTEGRAL

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In this note we give a proof of the theorem on integration by parts using the standard definition (see [2] or [3]) of the Perron integral in terms of major and minor functions.[1]

THEOREM. *Let f be a Perron integrable function and G a function of bounded variation on the finite interval $[a, b]$. Let*

$$F(x) = F(a) + (P)\int_a^x f\,dt, \qquad a \leqq x \leqq b.$$

Then fG is Perron integrable on $[a, b]$ and

$$(P)\int_a^b fG\,dx = F(x)G(x)\Big]_a^b - \int_a^b F\,dG, \tag{1}$$

the last integral being Riemann-Stieltjes.

We may assume that $G(a) = F(a) = 0$. Since every function G of bounded variation vanishing at $x = a$ can be written as a linear combination of nondecreasing continuous functions and nondecreasing jump functions vanishing at $x = a$, it is sufficient to prove the theorem for these two types of functions. This is done in Lemmas 3 and 4 below.

LEMMA 1. *The theorem holds when G is a nondecreasing jump function with a finite number of jumps and $G(a) = 0$.*

PROOF. Since G is the sum of a finite number of nondecreasing jump functions each having a single jump and vanishing at $x = a$, and since the lemma holds for each summand, it also holds for G.

LEMMA 2. *Let f be a Perron integrable function on the finite interval $[a, b]$. Then there is a constant k such that for every bounded nondecreas-*

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[1] It seems curious that a direct proof of the formula based only on the standard definition of the Perron integral does not exist in the literature (see the comment in [2, p. 101]). A proof based upon a definition of the Perron integral involving right and left major and minor functions is given in [4]. Proofs based upon the constructive and the descriptive definition of the special Denjoy integral—an equivalent of the Perron integral—are given respectively in [1] and [5].

ing function G on $[a, b]$ with $G(a)=0$, there is a major function and a minor function of fG both of which are bounded in absolute value by $kG(b)$.

PROOF. Let k be a constant greater than $\operatorname{Osc}(F)$, the oscillation on $[a, b]$ of F, the indefinite Perron integral of f. Let ψ be a major function of f with $\operatorname{Osc}(\psi) \leqq k$. For $a \leqq u \leqq b$, let

$$(2) \qquad \begin{aligned} \psi_u(x) &= \min[\psi(y): x \leqq y \leqq u] && \text{if } a \leqq x \leqq u \\ &= \psi(x) && \text{if } u \leqq x \leqq b. \end{aligned}$$

We will show that

$$M(x) = \int_a^b [\psi_u(x) - \psi_u(a)]dG(u)$$

is a major function of fG and that

$$(3) \qquad |M(x)| \leqq kG(b), \qquad a \leqq x \leqq b.$$

Since $\psi_u(x)$ is (uniformly) continuous jointly in x and u, $M(x)$ is continuous. Also $M(a)=0$. Let $a \leqq s \leqq t \leqq b$. Then

$$\begin{aligned} M(t) - M(s) &= \left(\int_a^s + \int_s^t + \int_t^b\right) [\psi_u(t) - \psi_u(s)]dG(u) \\ &\geqq [\psi(t) - \psi(s)]G(t). \end{aligned}$$

Hence, for x $(a<x<b)$ a point of continuity of G, the lower derivate of M at x

$$\underline{D}M(x) \geqq G(x)\underline{D}\psi(x).$$

Thus

$$\underline{D}M(x) > -\infty \text{ nearly everywhere}$$

(that is, everywhere in $[a, b]$ with the possible exception of a denumerable subset of $[a, b]$); and

$$\underline{D}M(x) \geqq f(x)G(x) \text{ a.e.}$$

Also, since $\operatorname{Osc}(\psi_u) \leqq \operatorname{Osc}(\psi)$, (3) holds.

In parallel fashion, replacing in (2), ψ, ψ_u, min, respectively by ϕ, ϕ_u, max, where ϕ is a minor function of f with $\operatorname{Osc}(\phi) \leqq k$, we obtain a minor function m of fG satisfying

$$|m(x)| \leqq kG(b), \qquad a \leqq x \leqq b.$$

LEMMA 3. *The theorem holds when G is a nondecreasing bounded jump function such that $G(a)=0$.*

PROOF. We may write $G=G_1+G_2$ with G_i $(i=1, 2)$ a nondecreasing bounded jump function such that $G_i(a)=0$, G_1 having a finite number of jumps, and $G_2(b)$ arbitrarily small. By virtue of Lemmas 1 and 2, there is a major function M and a minor function m of fG such that $M(b)-m(b)$ is arbitrarily small. Hence fG is Perron integrable. It follows from Lemma 2 that

$$\left|(P)\int_a^b fGdx\right| \leqq kG(b), \tag{4}$$

k depending only upon f. Using (by Lemma 1) equation (1) for G_1, and inequality (4) for G_2, we see that

$$(P)\int_a^b fGdx - F(b)G(b) + \int_a^b FdG$$

is arbitrarily small and hence equal to zero.

LEMMA 4. *The theorem holds for G continuous and nondecreasing, and* $G(a)=F(a)=0$.

PROOF. Let ψ and ϕ respectively be a major and a minor function of f and let

$$M(x) = \psi(x)G(x) - \int_a^x \phi dG. \tag{5}$$

Then $M=M(x)$ is[2] a major function of fG.

To begin with, M is continuous and $M(a)=0$. Let $a<x<b$. Let $t>0$. We get

$$\begin{aligned} M(x+t) - M(x) = {} & G(x+t)[\psi(x+t) - \psi(x)] \\ & + [\psi(x) - \phi(x)][G(x+t) - G(x)] \\ & - \int_x^{x+t} [\phi(u) - \phi(x)]dG(u), \end{aligned}$$

and

$$\begin{aligned} \frac{M(x+t) - M(x)}{t} \geqq {} & G(x+t)\frac{\psi(x+t) - \psi(x)}{t} \\ & - \int_x^{x+t} \frac{\phi(u) - \phi(x)}{u - x}\left(\frac{u - x}{t}\right)dG(u). \end{aligned}$$

Thus, the lower right derivate of M at x

[2] This form of M was suggested by A. Zygmund.

(6) $$D^+M(x) > -\infty \text{ nearly everywhere.}$$

Similarly for $t<0$,

$$\frac{M(x+t)-M(x)}{t} = G(x+t)\left[\frac{\psi(x+t)-\psi(x)}{t}\right] + \int_{x+t}^{x}\frac{\psi(u)-\psi(x)}{u-x}\left(\frac{u-x}{t}\right)dG(u) - \frac{1}{t}\int_{x+t}^{x}[\psi(u)-\phi(u)]dG(u);$$

and so the lower left derivate of M at x

(7) $$D^-M(x) > -\infty \text{ nearly everywhere.}$$

It follows from (6) and (7) that $DM(x) > -\infty$ nearly everywhere. Finally (as in [**4**]), for $t\neq 0$

$$\begin{aligned}\frac{M(x+t)-M(x)}{t} &= G(x)\left[\frac{\psi(x+t)-\psi(x)}{t}\right] \\ &\quad + \frac{1}{t}\int_{x}^{x+t}[\psi(x+t)-\psi(u)]dG(u) \\ &\quad + \frac{1}{t}\int_{x}^{x+t}[\psi(u)-\phi(u)]dG(u) \\ &= A+B+C.\end{aligned}$$

Now $C\geqq 0$. If x is a point where G has a finite derivative, then

$$B = [\psi(x+t)-\psi(x_0)]\frac{G(x+t)-G(x)}{t}\to 0$$

as $t\to 0$, x_0 being some point between x and $x+t$. Therefore

$$DM(x) \geqq G(x)f(x) \text{ a.e.}$$

Thus, M is a major function of fG. Similarly it can be shown that

$$m(x) = \phi(x)G(x) - \int_a^x \psi dG$$

is a minor function of fG. Now

$$M(b)-m(b) = [\psi(b)-\phi(b)]G(b) + \int_a^b(\psi-\phi)dG$$

is small when $\psi(b)-\phi(b)$ is small. Hence fG is Perron integrable; and letting ψ and ϕ uniformly approach F, (1) follows from (5).

References

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