AN ELEMENTARY PROOF OF INTEGRATION BY PARTS FOR THE PERRON INTEGRAL

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In this note we give a proof of the theorem on integration by parts using the standard definition (see [2] or [3]) of the Perron integral in terms of major and minor functions.1

THEOREM. Let $f$ be a Perron integrable function and $G$ a function of bounded variation on the finite interval $[a, b]$. Let

$$F(x) = F(a) + (P) \int_{a}^{x} f dt, \quad a \leq x \leq b.$$ 

Then $fG$ is Perron integrable on $[a, b]$ and

$$\int_{a}^{b} (P) fG dx = F(x)G(x) - \int_{a}^{b} F dG,$$ 

the last integral being Riemann-Stieltjes.

We may assume that $G(a) = F(a) = 0$. Since every function $G$ of bounded variation vanishing at $x = a$ can be written as a linear combination of nondecreasing continuous functions and nondecreasing jump functions vanishing at $x = a$, it is sufficient to prove the theorem for these two types of functions. This is done in Lemmas 3 and 4 below.

**Lemma 1.** The theorem holds when $G$ is a nondecreasing jump function with a finite number of jumps and $G(a) = 0$.

**Proof.** Since $G$ is the sum of a finite number of nondecreasing jump functions each having a single jump and vanishing at $x = a$, and since the lemma holds for each summand, it also holds for $G$.

**Lemma 2.** Let $f$ be a Perron integrable function on the finite interval $[a, b]$. Then there is a constant $k$ such that for every bounded nondecreasing

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1 It seems curious that a direct proof of the formula based only on the standard definition of the Perron integral does not exist in the literature (see the comment in [2, p. 101]). A proof based upon a definition of the Perron integral involving right and left major and minor functions is given in [4]. Proofs based upon the constructive and the descriptive definition of the special Denjoy integral—an equivalent of the Perron integral—are given respectively in [1] and [5].

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ing function $G$ on $[a, b]$ with $G(a) = 0$, there is a major function and a minor function of $fG$ both of which are bounded in absolute value by $kG(b)$.

**Proof.** Let $k$ be a constant greater than $\text{Osc}(F)$, the oscillation on $[a, b]$ of $F$, the indefinite Perron integral of $f$. Let $\psi$ be a major function of $f$ with $\text{Osc}(\psi) \leq k$. For $a \leq u \leq b$, let

$$\psi_u(x) = \min \{\psi(y) : x \leq y \leq u\} \quad \text{if } a \leq x \leq u$$

$$= \psi(x) \quad \text{if } u \leq x \leq b.$$  

(2)

We will show that

$$M(x) = \int_a^b [\psi_u(x) - \psi_u(a)] dG(u)$$

is a major function of $fG$ and that

$$|M(x)| \leq kG(b), \quad a \leq x \leq b.$$  

(3)

Since $\psi_u(x)$ is (uniformly) continuous jointly in $x$ and $u$, $M(x)$ is continuous. Also $M(a) = 0$. Let $a \leq s \leq t \leq b$. Then

$$M(t) - M(s) = \left(\int_a^s + \int_s^t + \int_t^b\right) [\psi_u(t) - \psi_u(s)] dG(u)$$

$$\geq [\psi(t) - \psi(s)] G(t).$$

Hence, for $x \ (a < x < b)$ a point of continuity of $G$, the lower derivate of $M$ at $x$

$$DM(x) \geq G(x) \psi(x).$$

Thus

$$DM(x) > -\infty \quad \text{nearly everywhere}$$

(that is, everywhere in $[a, b]$ with the possible exception of a denumerable subset of $[a, b]$); and

$$DM(x) \geq f(x)G(x) \text{ a.e.}$$

Also, since $\text{Osc}(\psi_u) \leq \text{Osc}(\psi)$, (3) holds.

In parallel fashion, replacing in (2), $\psi, \psi_u, \min$, respectively by $\phi, \phi_u, \max$, where $\phi$ is a minor function of $f$ with $\text{Osc}(\phi) \leq k$, we obtain a minor function $m$ of $fG$ satisfying

$$|m(x)| \leq kG(b), \quad a \leq x \leq b.$$  

**Lemma 3.** The theorem holds when $G$ is a nondecreasing bounded jump function such that $G(a) = 0$. 
Proof. We may write \( G = G_1 + G_2 \) with \( G_i \) \((i = 1, 2)\) a nondecreasing bounded jump function such that \( G_i(a) = 0 \), \( G_1 \) having a finite number of jumps, and \( G_2(b) \) arbitrarily small. By virtue of Lemmas 1 and 2, there is a major function \( M \) and a minor function \( m \) of \( fG \) such that \( M(b) - m(b) \) is arbitrarily small. Hence \( fG \) is Perron integrable. It follows from Lemma 2 that

\[
(P) \int_a^b fG dx \leq kG(b),
\]

\( k \) depending only upon \( f \). Using (by Lemma 1) equation (1) for \( G_1 \), and inequality (4) for \( G_2 \), we see that

\[
(P) \int_a^b fG dx - F(b)G(b) + \int_a^b FdG
\]

is arbitrarily small and hence equal to zero.

Lemma 4. The theorem holds for \( G \) continuous and nondecreasing, and \( G(a) = F(a) = 0 \).

Proof. Let \( \psi \) and \( \phi \) respectively be a major and a minor function of \( f \) and let

\[
M(x) = \psi(x)G(x) - \int_x^z \phi dG.
\]

Then \( M = M(x) \) is \(^2\) a major function of \( fG \).

To begin with, \( M \) is continuous and \( M(a) = 0 \). Let \( a < x < b \). Let \( t > 0 \). We get

\[
M(x + t) - M(x) = G(x + t)[\psi(x + t) - \psi(x)] + [\psi(x) - \phi(x)][G(x + t) - G(x)] - \int_x^{x+t} [\phi(u) - \phi(x)]dG(u),
\]

and

\[
\frac{M(x + t) - M(x)}{t} \geq G(x + t) \frac{\psi(x + t) - \psi(x)}{t} - \int_x^{x+t} \frac{\phi(u) - \phi(x)}{u - x} \left( \frac{u - x}{t} \right) dG(u).
\]

Thus, the lower right derivate of \( M \) at \( x \)

\(^2\) This form of \( M \) was suggested by A. Zygmund.
(6) \[ D^+M(x) > -\infty \text{ nearly everywhere.} \]

Similarly for \( t < 0 \),
\[
\frac{M(x + t) - M(x)}{t} = G(x + t) \left[ \frac{\psi(x + t) - \psi(x)}{t} \right] \\
+ \int_{x+t}^{x} \frac{\psi(u) - \psi(x)}{u - x} \frac{(u - x)}{t} \ dG(u) \\
- \frac{1}{t} \int_{x+t}^{x} [\psi(u) - \phi(u)] dG(u);
\]

and so the lower left derivate of \( M \) at \( x \)

(7) \[ D^-M(x) > -\infty \text{ nearly everywhere.} \]

It follows from (6) and (7) that \( DM(x) > -\infty \) nearly everywhere. Finally (as in [4]), for \( t \neq 0 \)
\[
\frac{M(x + t) - M(x)}{t} = G(x) \left[ \frac{\psi(x + t) - \psi(x)}{t} \right] \\
+ \frac{1}{t} \int_{x}^{x+t} \left[ \psi(x + t) - \psi(u) \right] dG(u) \\
+ \frac{1}{t} \int_{x}^{x+t} \left[ \psi(u) - \phi(u) \right] dG(u) \\
= A + B + C.
\]

Now \( C \geq 0 \). If \( x \) is a point where \( G \) has a finite derivative, then
\[
B = \left[ \psi(x + t) - \psi(x_0) \right] \frac{G(x + t) - G(x)}{t} \to 0
\]
as \( t \to 0 \), \( x_0 \) being some point between \( x \) and \( x + t \). Therefore
\[ DM(x) \geq G(x) \phi(x) \text{ a.e.} \]

Thus, \( M \) is a major function of \( fG \). Similarly it can be shown that
\[
m(x) = \phi(x)G(x) - \int_{a}^{x} \psi dG
\]
is a minor function of \( fG \). Now
\[
M(b) - m(b) = [\psi(b) - \phi(b)]G(b) + \int_{a}^{b} (\psi - \phi) dG
\]
is small when $\psi(b) - \phi(b)$ is small. Hence $fG$ is Perron integrable; and letting $\psi$ and $\phi$ uniformly approach $F$, (1) follows from (5).

References


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