AN ELEMENTARY PROOF OF INTEGRATION BY PARTS FOR THE PERRON INTEGRAL

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In this note we give a proof of the theorem on integration by parts using the standard definition (see [2] or [3]) of the Perron integral in terms of major and minor functions.\(^1\)

**Theorem.** Let \( f \) be a Perron integrable function and \( G \) a function of bounded variation on the finite interval \([a, b]\). Let

\[
F(x) = F(a) + \left( P \right) \int_a^x f \, dt, \quad a \leq x \leq b.
\]

Then \( fG \) is Perron integrable on \([a, b]\) and

\[
\left( P \right) \int_a^b fG \, dx = F(x)G(x) \left[ b \right]_a^b - \int_a^b F \, dG,
\]

the last integral being Riemann-Stieltjes.

We may assume that \( G(a) = F(a) = 0 \). Since every function \( G \) of bounded variation vanishing at \( x = a \) can be written as a linear combination of nondecreasing continuous functions and nondecreasing jump functions vanishing at \( x = a \), it is sufficient to prove the theorem for these two types of functions. This is done in Lemmas 3 and 4 below.

**Lemma 1.** The theorem holds when \( G \) is a nondecreasing jump function with a finite number of jumps and \( G(a) = 0 \).

**Proof.** Since \( G \) is the sum of a finite number of nondecreasing jump functions each having a single jump and vanishing at \( x = a \), and since the lemma holds for each summand, it also holds for \( G \).

**Lemma 2.** Let \( f \) be a Perron integrable function on the finite interval \([a, b]\). Then there is a constant \( k \) such that for every bounded nondecreas-

\(^1\) It seems curious that a direct proof of the formula based only on the standard definition of the Perron integral does not exist in the literature (see the comment in [2, p. 101]). A proof based upon a definition of the Perron integral involving right and left major and minor functions is given in [4]. Proofs based upon the constructive and the descriptive definition of the special Denjoy integral—an equivalent of the Perron integral—are given respectively in [1] and [5].
Proof of Integration by Parts

Let \( k \) be a constant greater than \( \text{Osc}(F) \), the oscillation on \([a, b]\) of \( F \), the indefinite Perron integral of \( f \). Let \( \psi \) be a major function of \( f \) with \( \text{Osc}(\psi) \leq k \). For \( a \leq u \leq b \), let

\[
\psi_u(x) = \min\{\psi(y) : x \leq y \leq u\} \quad \text{if } a \leq x \leq u
\]
\[
= \psi(x) \quad \text{if } u \leq x \leq b.
\]

We will show that

\[
M(x) = \int_a^b [\psi_u(x) - \psi_u(a)]dG(u)
\]

is a major function of \( fG \) and that

\[
|M(x)| \leq kG(b), \quad a \leq x \leq b.
\]

Since \( \psi_u(x) \) is (uniformly) continuous jointly in \( x \) and \( u \), \( M(x) \) is continuous. Also \( M(a) = 0 \). Let \( a \leq s \leq t \leq b \). Then

\[
M(t) - M(s) = \left( \int_a^s + \int_s^t + \int_t^b \right) [\psi_u(t) - \psi_u(s)]dG(u)
\]
\[
\geq [\psi(t) - \psi(s)]G(t).
\]

Hence, for \( x \ (a < x < b) \) a point of continuity of \( G \), the lower derivate of \( M \) at \( x \)

\[
DM(x) \geq G(x)D\psi(x).
\]

Thus

\[
DM(x) > -\infty \text{ nearly everywhere}
\]

(that is, everywhere in \([a, b]\) with the possible exception of a denumerable subset of \([a, b]\)); and

\[
DM(x) \geq f(x)G(x) \text{ a.e.}
\]

Also, since \( \text{Osc}(\psi_u) \leq \text{Osc}(\psi) \), (3) holds.

In parallel fashion, replacing in (2), \( \psi, \psi_u, \min \), respectively by \( \phi \), \( \phi_u, \max \), where \( \phi \) is a minor function of \( f \) with \( \text{Osc}(\phi) \leq k \), we obtain a minor function \( m \) of \( fG \) satisfying

\[
|m(x)| \leq kG(b), \quad a \leq x \leq b.
\]

Lemma 3. The theorem holds when \( G \) is a nondecreasing bounded jump function such that \( G(a) = 0 \).
Proof. We may write $G = G_1 + G_2$ with $G_i$ ($i = 1, 2$) a nondecreasing bounded jump function such that $G_i(a) = 0$, $G_1$ having a finite number of jumps, and $G_2(b)$ arbitrarily small. By virtue of Lemmas 1 and 2, there is a major function $M$ and a minor function $m$ of $fG$ such that $M(b) - m(b)$ is arbitrarily small. Hence $fG$ is Perron integrable. It follows from Lemma 2 that

\[ (P) \int_a^b fGdx \leq kG(b), \]

$k$ depending only upon $f$. Using (by Lemma 1) equation (1) for $G_1$, and inequality (4) for $G_2$, we see that

\[ (P) \int_a^b fGdx - F(b)G(b) + \int_a^b FdG \]

is arbitrarily small and hence equal to zero.

Lemma 4. The theorem holds for $G$ continuous and nondecreasing, and $G(a) = F(a) = 0$.

Proof. Let $\psi$ and $\phi$ respectively be a major and a minor function of $f$ and let

\[ M(x) = \psi(x)G(x) - \int_a^x \phi dG. \]

Then $M = M(x)$ is a major function of $fG$.

To begin with, $M$ is continuous and $M(a) = 0$. Let $a < x < b$. Let $\epsilon > 0$. We get

\[
M(x + \epsilon) - M(x) = G(x + \epsilon) [\psi(x + \epsilon) - \psi(x)]
+ [\psi(x) - \phi(x)] [G(x + \epsilon) - G(x)]
- \int_x^{x + \epsilon} [\phi(u) - \phi(x)] dG(u),
\]

and

\[
\frac{M(x + \epsilon) - M(x)}{\epsilon} \geq G(x + \epsilon) \frac{\psi(x + \epsilon) - \psi(x)}{\epsilon}
- \int_x^{x + \epsilon} \frac{\phi(u) - \phi(x)}{u - x} \left(\frac{u - x}{\epsilon}\right) dG(u).
\]

Thus, the lower right derivate of $M$ at $x$

\(^2\) This form of $M$ was suggested by A. Zygmund.
Proof of Integration by Parts

(6) \[ D^+M(x) > -\infty \text{ nearly everywhere.} \]

Similarly for \( t < 0 \),

\[
\frac{M(x + t) - M(x)}{t} = G(x + t) \left[ \frac{\psi(x + t) - \psi(x)}{t} \right] \\
+ \int_{x+t}^{\infty} \frac{\psi(u) - \psi(x)}{u - x} \left( \frac{u - x}{t} \right) dG(u) \\
- \frac{1}{t} \int_{x+t}^{\infty} [\psi(u) - \phi(u)] dG(u);
\]

and so the lower left derivative of \( M \) at \( x \)

(7) \[ D^-M(x) > -\infty \text{ nearly everywhere.} \]

It follows from (6) and (7) that \( DM(x) > -\infty \) nearly everywhere. Finally (as in [4]), for \( t \neq 0 \)

\[
\frac{M(x + t) - M(x)}{t} = G(x) \left[ \frac{\psi(x + t) - \psi(x)}{t} \right] \\
+ \frac{1}{t} \int_{x}^{x+t} [\psi(x + t) - \psi(u)] dG(u) \\
+ \frac{1}{t} \int_{x}^{x+t} [\psi(u) - \phi(u)] dG(u) \\
= A + B + C.
\]

Now \( C \geq 0 \). If \( x \) is a point where \( G \) has a finite derivative, then

\[
B = [\psi(x + t) - \psi(x_0)] \frac{G(x + t) - G(x)}{t} \to 0
\]

as \( t \to 0 \), \( x_0 \) being some point between \( x \) and \( x + t \). Therefore

\[ DM(x) \geq G(x)f(x) \text{ a.e.} \]

Thus, \( M \) is a major function of \( fG \). Similarly it can be shown that

\[ m(x) = \phi(x)G(x) - \int_{a}^{x} \psi dG \]

is a minor function of \( fG \). Now

\[
M(b) - m(b) = [\psi(b) - \phi(b)]G(b) + \int_{a}^{b} (\psi - \phi) dG
\]
is small when \( \psi(b) - \phi(b) \) is small. Hence \( fG \) is Perron integrable; and letting \( \psi \) and \( \phi \) uniformly approach \( F \), (1) follows from (5).

References


