EQUATIONS EQUIVALENT TO NONLINEAR DIFFERENTIAL EQUATIONS

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1.0. Introduction. The solution of a class of ordinary nonlinear differential equations can be established in terms of the solution of a linear ordinary differential equation; this linear base equation being readily identified from the nonlinear equation. Pinney [1] found that the solution of the specific nonlinear differential equation, $y''+p(x)y'+cy^{-3}=0$, c constant, can be expressed as $y=(u_1^2-u_2^2)^{1/2}$, where $u_1(x)$, $u_2(x)$ form a fundamental set of solutions to $u''+p(x)u=0$. Thomas [2] posed the question: Which nth order equations have general solutions expressible in the form $y=F(u_1, \cdots, u_n)$ where $u_1, \cdots, u_n$ constitute an appropriate set of solutions of a linear equation? Thomas answered the question in terms of the first order linear base equation $u'+p(x)u+q(x)=0$ and also in terms of the homogeneous second order equation, $u''+p(x)u'+q(x)u=0$ where in the second order form, F depends on either one variable, u, or F is homogeneous of nonzero degree in two u's. Herbst [3] and Gergen and Dressel [4] have extended Thomas' formulation for F in two variables.

The work of the previous investigators suggests the question: Which nonlinear differential equations can be solved in terms of the solutions to other, specific, nonlinear differential equations? This question is answered for the second order elliptic equation $u''=au+bu^3$, a, b, constants, the results being given in Theorem 1.

THEOREM 1. *If u is a solution of the elliptic equation $u''=au+bu^3$ where a, b are constants, then the equation $y''=f(y, y', a, b)$, $(')=d/dx$, has the general solution $y=F(u)$ if and only if $f=aZ(y)+bT(y)+A(y)y'^2$ where Z, T, A satisfy $\dot{Z}-AZ=1$, $Z\dot{T}-(3+AZ)T=0$, $(\dot{\ })=d/dy$. The function F is found from the relationship $u^2=T(F)/Z(F)$.*

In addition, the work of Thomas is extended to include the second order nonhomogeneous linear base equation $u''+p(x)u'+q(x)u+r(x)=0$ where the function F is a function of one variable, u. In this extension, and also for his first order formulation, integrals of the functional relations are found.

2.0. Nonlinear equations having nonlinear bases. This section

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presents the derivation of the results given in Theorem 1 and some of its consequences.

If u is a solution of the elliptic equation

$$u'' = au + bu^3 \tag{2-1}$$

and it is assumed that the derivable nonlinear equation has the solution $y = F(u)$, then

$$y'' = F_u(au + bu^3) + F_{uu}u'^2. \tag{2-2}$$

Selecting the functional relation

$$F_u = g(y) \tag{2-3}$$

with its solution in terms of u

$$u = \int^y g(s)^{-1}ds$$

then the equation (2-2) becomes a function of y and x

$$y'' = ag(y)\int^y g(s)^{-1}ds + bg(y)\left[\int^y g(s)^{-1}ds\right]^3 + g(y)^{-1}\dot{g}(y)y'^2. \tag{2-4}$$

Rewriting (2-4), the general second order nonlinear differential equation having (2-1) as its base equation is

$$y'' = aZ(y) + bT(y) + A(y)y'^2 \tag{2-5}$$

where Z, T, A satisfy,

$$\dot{Z} - AZ = 1, \qquad Z\dot{T} - (3 + AZ)T = 0. \tag{2-6}$$

Finally, the function F can be found from the integral of (2-3) as

$$u^2 = T(F)/Z(F). \tag{2-7}$$

Let F be determined from (2-7) and let $y = F(u)$ be a solution of (2-5) where Z, T, A, satisfy (2-6). In order to show that u is a solution of the elliptic equation, it is necessary to form y'' using (2-6) and (2-7). Once this is accomplished, it follows that u satisfies (2-1).

If initial values y_0, y_0' are given, then a corresponding set of initial values u_0, u_0' can be found using $y = F(u)$. Thus $y = F(u)$ is a general solution of (2-5) and Theorem 1 is established.

An implication of Theorem 1 is that any specific solvable nonlinear equation can be used as the base equation. This suggests that the specific nonlinear base equation could be of the class having a solution in terms of solutions to linear equations. Thus the solution of the general nonlinear equations should be obtainable by means of a double transformation. It is easily demonstrated that this is indeed the case. However, as the general nonlinear equation does have a linear base, it will be directly solvable by the methods associated with linear base nonlinear equations. Hence, no new nonlinear equations will be generated by this double transformation.

3.0. **Nonlinear equations with linear base equations.**

3.1. *First order equations.* In this section it is shown that the function F follows directly from Thomas' results. Thomas has shown [2, §2] that the general first order equation

$$y' + p(x)g(y) + q(x)f(y) = 0$$

where g, f satisfy either

$$f(\dot{g}/f) = 1 \quad \text{or} \quad g(\dot{f}/g) = 1. \tag{3-1}$$

Then the functional relation

$$F_u - f(F) = 0 \quad \text{or} \quad F_u - g(F) = 0 \tag{3-2}$$

yields a solution in the form $y = F(u)$ where u is a solution of the base equation

$$u' + p(x)u + q(x) = 0 \quad \text{or} \quad u' + q(x)u + p(x) = 0.$$

However, Thomas neglected to show that F can be determined directly from (3-1) and (3-2) in terms of u:

$$u = g(F)/f(F) \quad \text{or} \quad u = f(F)/g(F).$$

3.2 *Second order equations, F in one variable.* In this section the results of Thomas [2, §3] are extended to establish the solutions of the class of nonlinear equations having a linear nonhomogeneous base equation. His results are directly obtainable from those presented here.

The relations of this section form a theorem analogous to Theorem 1 and has a proof of a similar nature.

Consider the nonhomogeneous linear differential equation

$$u'' + p(x)u' + q(x)u + r(x) = 0. \tag{3-3}$$

If $y = F(u)$, y being the solution of the derivable nonlinear differential equation, the following second order identity can be written

$$y'' + p(x)y' = F_u[-q(x)u - r(x)] + F_{uu}u'^2. \tag{3-4}$$

Choosing the functional relation as

$$F_u = Z(y), \qquad u = \int^y Z(s)^{-1} ds$$

the identity (3-4) can be written as:

$$y'' + p(x)y' + q(x)g(y) + r(x)Z(y) = Z^{-1}\dot{Z}y'^2 \tag{3-5}$$

where g, Z satisfy

$$Z(\dot{g}/Z)^{\cdot} = 1 \tag{3-6}$$

which, using (3-6), has the integral

$$u = g(F)/Z(F). \tag{3-7}$$

Let $y = F(u)$, F determined from (3-7), be a solution of (3-5) satisfying (3-6). Then u satisfies the base equation (3-3).

The general form of the nonlinear equation is (3-5). If and only if Z, g satisfy (3-6), then $y = F(u)$ is a general solution where F is determined from (3-7) and u satisfies (3-3). Finally, using (3-6) and the coefficient of y'^2, an equivalence can be established with Thomas' results [2, equations (3.1), (3.2)] with the added integral of the functional relation.

References

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