EQUATIONS EQUIVALENT TO NONLINEAR DIFFERENTIAL EQUATIONS

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1.0. Introduction. The solution of a class of ordinary nonlinear differential equations can be established in terms of the solution of a linear ordinary differential equation; this linear base equation being readily identified from the nonlinear equation. Pinney [1] found that the solution of the specific nonlinear differential equation, \( y'' + p(x)y' + cy^{-3} = 0 \), \( c \) constant, can be expressed as \( y = (u_1^2 - u_2^2)^{1/2} \), where \( u_1(x), u_2(x) \) form a fundamental set of solutions to \( u'' + p(x)u = 0 \).

Thomas [2] posed the question: Which \( n \)th order equations have general solutions expressible in the form \( y = F(u_1, \cdots, u_n) \) where \( u_1, \cdots, u_n \) constitute an appropriate set of solutions of a linear equation? Thomas answered the question in terms of the first order linear base equation \( u' + p(x)u + q(x) = 0 \) and also in terms of the homogeneous second order equation, \( u'' + p(x)u' + q(x)u = 0 \) where in the second order form, \( F \) depends on either one variable, \( u \), or \( F \) is homogeneous of nonzero degree in two \( u \)'s. Herbst [3] and Gergen and Dressel [4] have extended Thomas' formulation for \( F \) in two variables.

The work of the previous investigators suggests the question: Which nonlinear differential equations can be solved in terms of the solutions to other, specific, nonlinear differential equations? This question is answered for the second order elliptic equation \( u'' = au + bu^3 \), \( a, b \), constants, the results being given in Theorem 1.

**Theorem 1.** If \( u \) is a solution of the elliptic equation \( u'' = au + bu^3 \) where \( a, b \) are constants, then the equation \( y'' = f(y, y', a, b) \), \( (') = d/dx \), has the general solution \( y = F(u) \) if and only if \( f = aZ(y) + bT(y) + A(y)y'^2 \) where \( Z, T, A \) satisfy \( \dot{Z} - AZ = 1 \), \( Z\dot{T} - (3 + AZ)T = 0 \), \( (\cdot) = d/dy \). The function \( F \) is found from the relationship \( u^2 = T(F)/Z(F) \).

In addition, the work of Thomas is extended to include the second order nonhomogeneous linear base equation \( u'' + p(x)u' + q(x)u + r(x) = 0 \) where the function \( F \) is a function of one variable, \( u \). In this extension, and also for his first order formulation, integrals of the functional relations are found.

2.0. Nonlinear equations having nonlinear bases. This section

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presents the derivation of the results given in Theorem 1 and some of its consequences.

If \( u \) is a solution of the elliptic equation
\[
(2-1) \quad u'' = au + bu^3
\]
and it is assumed that the derivable nonlinear equation has the solution \( y = F(u) \), then
\[
(2-2) \quad y'' = F_u(au + bu^3) + F_{uu}u'^2.
\]
Selecting the functional relation
\[
(2-3) \quad F_u = g(y)
\]
with its solution in terms of \( u \)
\[
\int g(s)^{-1} ds
\]
then the equation (2-2) becomes a function of \( y \) and \( x \)
\[
(2-4) \quad y'' = ag(y) \int g(s)^{-1} ds + bg(y) \left[ \int g(s)^{-1} ds \right]^3 + g(y)^{-1}g'(y)y'^2.
\]
Rewriting (2-4), the general second order nonlinear differential equation having (2-1) as its base equation is
\[
(2-5) \quad y'' = aZ(y) + bT(y) + A(y)y'^2
\]
where \( Z, T, A \) satisfy,
\[
(2-6) \quad \dot{Z} - AZ = 1, \quad Z\dot{T} - (3 + AZ)T = 0.
\]
Finally, the function \( F \) can be found from the integral of (2-3) as
\[
(2-7) \quad u^2 = T(F)/Z(F).
\]

Let \( F \) be determined from (2-7) and let \( y = F(u) \) be a solution of (2-5) where \( Z, T, A \), satisfy (2-6). In order to show that \( u \) is a solution of the elliptic equation, it is necessary to form \( y'' \) using (2-6) and (2-7). Once this is accomplished, it follows that \( u \) satisfies (2-1).

If initial values \( y_0, y'_0 \) are given, then a corresponding set of initial values \( u_0, u'_0 \) can be found using \( y = F(u) \). Thus \( y = F(u) \) is a general solution of (2-5) and Theorem 1 is established.

An implication of Theorem 1 is that any specific solvable nonlinear equation can be used as the base equation. This suggests that the specific nonlinear base equation could be of the class having a solution in terms of solutions to linear equations. Thus the solution of the
general nonlinear equations should be obtainable by means of a
double transformation. It is easily demonstrated that this is indeed
the case. However, as the general nonlinear equation does have a
linear base, it will be directly solvable by the methods associated with
linear base nonlinear equations. Hence, no new nonlinear equations
will be generated by this double transformation.

3.0. **Nonlinear equations with linear base equations.**

3.1. **First order equations.** In this section it is shown that the func-
tion $F$ follows directly from Thomas' results. Thomas has shown [2,
§2] that the general first order equation

$$y' + p(x)g(y) + q(x)f(y) = 0$$

where $g, f$ satisfy either

$$f(g/f) = 1 \quad \text{or} \quad g(f/g) = 1.$$  

Then the functional relation

$$F_u - f(F) = 0 \quad \text{or} \quad F_u - g(F) = 0$$

yields a solution in the form $y = F(u)$ where $u$ is a solution of the base
equation

$$u' + p(x)u + q(x) = 0 \quad \text{or} \quad u' + q(x)u + p(x) = 0.$$  

However, Thomas neglected to show that $F$ can be determined
directly from (3-1) and (3-2) in terms of $u$:

$$u = g(F)/f(F) \quad \text{or} \quad u = f(F)/g(F).$$

3.2 **Second order equations, $F$ in one variable.** In this section the
results of Thomas [2, §3] are extended to establish the solutions of the
class of nonlinear equations having a linear nonhomogeneous base
equation. His results are directly obtainable from those presented
here.

The relations of this section form a theorem analogous to Theorem
1 and has a proof of a similar nature.

Consider the nonhomogeneous linear differential equation

$$u'' + p(x)u' + q(x)u + r(x) = 0.$$  

If $y = F(u)$, $y$ being the solution of the derivable nonlinear differential
equation, the following second order identity can be written

$$y'' + p(x)y' = F_u[-q(x)u - r(x)] + F_{uu}u'^2.$$  

Choosing the functional relation as
\[ F_u = Z(y), \quad u = \int^y Z(s)^{-1} ds \]

the identity (3-4) can be written as:

\[ y'' + p(x)y' + q(x)g(y) + r(x)Z(y) = Z^{-1}Zy'^2 \]

where \( g, Z \) satisfy

\[ Z(g/Z) = 1 \]

which, using (3-6), has the integral

\[ u = g(F)/Z(F). \]

Let \( y = F(u) \), \( F \) determined from (3-7), be a solution of (3-5) satisfying (3-6). Then \( u \) satisfies the base equation (3-3).

The general form of the nonlinear equation is (3-5). If and only if \( Z, g \) satisfy (3-6), then \( y = F(u) \) is a general solution where \( F \) is determined from (3-7) and \( u \) satisfies (3-3). Finally, using (3-6) and the coefficient of \( y'^2 \), an equivalence can be established with Thomas' results [2, equations (3.1), (3.2)] with the added integral of the functional relation.

**References**


**Hamilton Standard, Windsor Locks, Connecticut**