ON ITERATES OF CONTINUOUS FUNCTIONS ON A UNIT BALL

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Let B^n be the unit ball in R^n, Euclidean n-space, i.e. $B^n = \{x: x \in R^n, d(x, 0) \leqq 1\}$. If f and g are any two functions of B^n to itself define, as usual, $\|f-g\| = \sup\{d(f(x), g(x)): x \in B^n\}$. J. Ax [1] has conjectured that if f is a continuous function of B^n onto itself such that f restricted to the boundary is the identity, then $\|f^{(m+1)}-I\| \geqq \|f^{(m)}-I\|$ for $m=1, 2, \cdots$, where I is the identity and $f^{(m)}$ is the mth iterate of f, e.g. $f(f(x)) = f^{(2)}(x)$.

Theorem 1 below shows the conjecture is true for $n=1$. Theorem 2 shows the conjecture is false for $n=2$. A concluding comment disposes of the cases $n>2$.

THEOREM 1. *Let f be a continuous function on $[-1, 1]$ to itself such that $f(-1)=-1$ and $f(1)=1$. Then $\|f^{(m+1)}-I\| \geqq \|f^{(m)}-I\|$.*

PROOF. The theorem is trivially true when f is the identity so we will only consider the case where $f \neq I$. The proof is by induction.

Let $G = \{x: f(x) = x\}$. The complement of G is a collection of disjoint open subintervals S_α of $[-1, 1]$. On each S_α, either $f(x) > x$ or $f(x) < x$ for all $x \in S_\alpha$. If we let $f^{(0)} = I$, the first step of the induction is clear. (One could start out at $f^{(1)} = f$, but it is not necessary.) Assume then that for $k = 1, 2, 3, \cdots, m$ we have $\|f^{(k)} - I\| > \|f^{(k-1)} - I\|$. Since $[-1, 1]$ is compact, we know there exists $r \in [-1, 1]$ such that $|f^{(m)}(r) - r| = \|f^{(m)} - I\|$. We assume $f^{(m)}(r) > r$ ($f^{(m)}(r) < r$ is argued analogously) and show $f(r) > r$. It is clear that $f(r) \neq r$. If $f(r) < r$ then for $s = f(r)$ we have

$$\begin{aligned} \|f^{(m-1)} - I\| &\geqq |f^{(m-1)}(s) - s| \\ &= |f^{(m)}(r) - s| = f^{(m)}(r) - s \\ &> f^{(m)}(r) - r = \|f^{(m)} - I\|, \end{aligned}$$

a contradiction. Hence $r \in S_\alpha$ on which $f(x) > x$ for all $x \in S_\alpha$. If $S_\alpha = (a, b)$ then $f(a) = a < r < b = f(b)$, and the continuity of f insures the existence of $q \in S_\alpha$ such that $f(q) = r$. Moreover, $q < r$ since $f(x) > x$ on S_α. Now,

$$\begin{aligned} \|f^{(m+1)} - I\| &\geqq |f^{(m+1)}(q) - q| \\ &= f^{(m)}(r) - q > f^{(m)}(r) - r = \|f^{(m)} - I\| \end{aligned}$$

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which completes the proof. We note that if f is not the identity we have actually shown that $\|f^{(m)}-I\|>\|f^{(m-1)}-I\|$.

THEOREM II. *Let $n\geqq 2$. Then there exist certain maps satisfying the conditions in the initial paragraph for which $\|f^{(k)}-I\|<\|f-I\|$.*

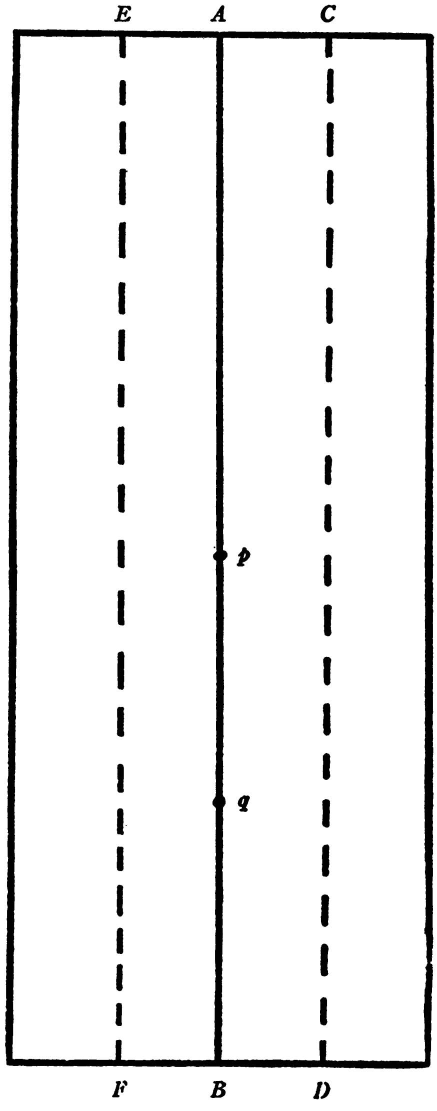

FIGURE 1

PROOF. Consider a rectangular strip $S=\{(x, y): -a\leqq x\leqq a, 0\leqq y\leqq 1\}$ (Figure I) where a is some positive number. Let p be the point in S having co-ordinates (0, 1/2) and $q\in S$ having co-ordinates (0, 1/4). These choices are arbitrary but they help fix ideas. Let AB, CD, EF be the vertical lines in S such that $x=0$, $a/2$, and $-a/2$ respectively. Let $g: S\rightarrow S$ where $g(x, y)=(x, y^{2-|x|/a})$. This defines a flow which is continuous (even C^∞), which is the identity on the boundary, and where all interior points flow downward on a vertical line.

Now consider a plane set Z illustrated in Figure II, and let the dis-

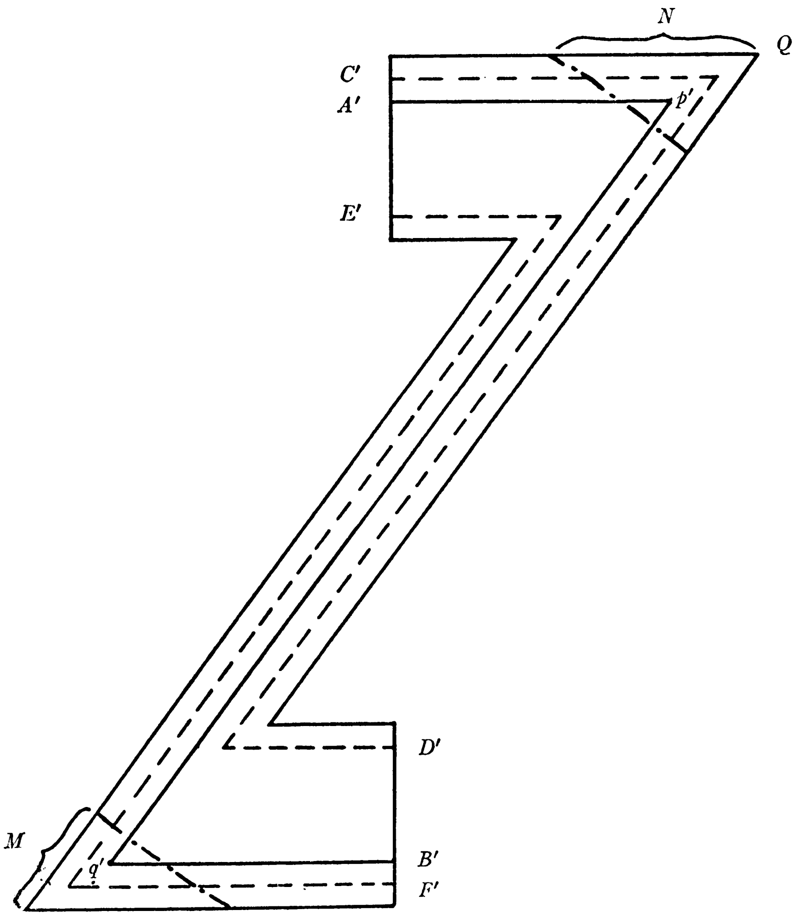

FIGURE II

tance between the points labeled Q and R be r. Choose a conveniently small positive number ϵ, and define N to be the set of all points in Z whose distance from R is greater than $r-\epsilon$, and define M as the set of points in Z having distance from Q greater than $r-\epsilon$. Clearly if $x, y \in Z$ and $d(x, y) > r-\epsilon$ then $x \in N$ and $y \in M$, or conversely. Select p' interior to N and q' interior to M so that $d(p', q') > r-\epsilon$. Now, let h be a homeomorphism of S onto Z taking $p \to p'$ and $q \to q'$ and the lines AB, CD, EF, respectively onto the broken lines $A'B'$, $C'D'$, and $E'F'$ as indicated in Figure II. Note, in particular, that the image of CD is disjoint from $\overline{M}$ while the image of EF misses $\overline{N}$, where the bar indicates the closure of the set.

Consider the set Z imbedded in the interior of B^2 and define $f: B^2 \to B^2$ by

$$f(x) = x \qquad \text{if } x \notin Z,$$
$$f(x) = hgh^{-1}(x) \qquad \text{if } x \in Z.$$

Since g restricted to the boundary of S is the identity, f restricted to the boundary of Z is the identity, and hence f is continuous—in fact, is a homeomorphism. Moreover, $\|f-I\| \geqq d(f(p'), p') = d(q', p') > r-\epsilon$. Now, notice that the image of CD cuts N into two components, say U and V, where U is the component containing p' and V the other. For $x \in U$, the sequence $\{f^{(i)}(x)\}$ converges to some point on the line segment $D'F'$; hence there is an integer k such that if $i \geqq k$, $f^{(i)}(x) \notin M$ for all $x \in U$. Now, since for $x \notin N$, $d(f^{(i)}(x), x) \leqq r-\epsilon$, and for $x \in V$, $d(f^{(i)}(x), x) < r-\epsilon$, and for $x \in U$, $d(f^{(k)}(x), x) \leqq r-\epsilon$, we have

$$\|f^{(k)} - I\| \leqq r - \epsilon < \|f - I\|.$$

Therefore, the sequence $\{\|f^{(m)} - I\|\}$ cannot be monotone increasing.

This proof can, of course, be extended to dimensions greater than 2 by applying f on the first two co-ordinates of a point and leaving the other co-ordinates fixed.

The reader should note that in the cases $n \geqq 2$ a 1-dimensional construction has been used. In fact every subset of B^n is homeomorphic to B^1 by a map which preserves order relations between distances if $n = 1$. This is not true for B^n, $n \geqq 2$, as the Z-shaped figure shows.

REFERENCE

1. J. Ax, Oral communication.

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