A SIMPLE PROOF OF A THEOREM OF ISBELL

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John Isbell [**3**, pp. 301–302] proved the following theorem.[1,2]

THEOREM. *Every complete metric space is homeomorphic with a closed subset of a countable product of finite-dimensional uniform complexes.*[3]

We shall give a simple proof of this theorem utilizing embeddings in Hilbert space.

Consider any real Hilbert space H. Let E be an orthonormal basis of H; let θ be the origin of H. Let $E' = E \cup \{\theta\}$. Let $|K|$ be the algebraic convex hull of E', whence $|K| \subset H$. We consider $|K|$ to be the underlying space of an obvious geometric simplicial complex K whose set of vertices is E'. Let $K^{(n)}$ be the n-skeleton of K; it is easily verified that $K^{(n)}$ as a uniform complex and $|K^{(n)}|$ as a topological subspace of H are topologically the same. Let Z be the closure of $|K|$ in H, whence Z is the set of all $v \in H$ such that $v = \sum_{e \in E} t_e e$ and $\sum_{e \in E} t_e \leqq 1$ for some nonnegative scalars t_e.

LEMMA 1. *Z is homeomorphic to a closed subset of $\prod_{n=0}^{\infty} |K^{(n)}|$.*

PROOF. Define $f_n: Z \to |K^{(n)}|$ as follows. If $v = \sum_{e \in E} t_e e$ as above, then, where s_e is the smaller of t_e and $(n+1)^{-1}$, let $f_n(v) = \sum_{e \in E} (t_e - s_e)e$. Define $f: Z \to \prod_{n=0}^{\infty} |K^{(n)}|$ so that f_n is the n-coordinate function of f for each n. It is easily shown that f is a homeomorphism from Z onto a closed subset of $\prod_{n=0}^{\infty} |K^{(n)}|$.

LEMMA 2. *Suppose that (X, d) is a metric space and the orthonormal dimension of H is as large as the cardinal of some base of X. Then there is a homeomorphism g from X into Z (defined as above) such that $g^{-1}: g[X] \to X$ is a uniformly continuous. Hence, if (X, d) is complete, then X is homeomorphic to a closed subset of Z (and thus to a closed subset of H).*

PROOF. After changing several obvious coefficients n^{-1} to (for

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[1] The question raised by the writer in his Zentralblatt (Vol. 119, pp. 181–182) review of [3] concerning the proof of this theorem arose from a notational misinterpretation; the proof in [3] is complete as it stands. The dimension Δd studied in [3] should have been attributed to [2] in the same review. The writer regrets these inaccuracies.

[2] Isbell [3] uses this theorem to obtain a much sharper result.

[3] A uniform complex is metrized by the maximum difference of barycentric coordinates.

example) 2^{-n} in [**1**, pp. 194–196], apply [**1**, 9.4, p. 196] and its proof. Thus Lemma 2 comes from scrutinizing a standard proof of the Nagata-Smirnov metrization theorem.

Finally, the theorem follows immediately from Lemmas 1 and 2.

References

1. J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1966.

2. J. R. Isbell, *Zero-dimensional spaces*, Tôhoku Math. J. (2) **7** (1955), 1–8.

3. ———, *On finite-dimensional uniform spaces*. II, Pacific J. Math. **12** (1962), 291–302.

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