KERNELS OF FREE ABELIAN REPRESENTATIONS OF A LINK GROUP

C. B. SCHAUFELE

1. **Introduction.** Let L denote an oriented link in the 3-sphere S^3; let $k_1, k_2, \cdots, k_\mu$ denote the components of L, $G=\pi_1(S^3-L)$ the group of the link. $H_1(S^3-L)$ is free abelian of rank μ, so we have the exact sequence

$$0 \to [G, G] \to G \to A_1 \oplus A_2 \oplus \cdots \oplus A_\mu \to 0$$

where $[G, G]$ is the commutator subgroup of G and each A_i, $i=1, \cdots, \mu$, is infinite cyclic. Thus we have homomorphisms

$$\phi_i \colon G \to A_i$$

for each $i=1, 2, \cdots, \mu$.

It is the purpose of this paper to determine the structure of the kernels of these homomorphisms. In order to do this, we will construct infinite cyclic covering spaces over S^3-L using methods similar to those of Lee Neuwirth in [2]. Using these infinite cyclic covering spaces, we will observe how to construct covering spaces over S^3-L corresponding to the kernels of homomorphisms $G \to A_{i_1} \oplus \cdots \oplus A_{i_m}$, $1 \leqq i_j \leqq \mu$, and, under certain conditions, it will be possible to determine the structure of these kernels, in particular, of the commutator subgroup of G.

2. **Construction of covering spaces.** Let S_i' be an orientable surface in S^3 which spans the oriented component k_i of L. We assume S_i' to be oriented so that $\partial S_i' = k_i$. S_i' is taken in general position relative to $L-k_i$ so that $k_j \cap S_i'$, $j \neq i$, is a finite set of points. These points are assigned a $+$ or $-$ in the usual manner. If $p_1, p_2 \in k_j \cap S_i'$ have different signs, and if one of the arcs $A \subset k_j$ from p_1 to p_2 does not intersect S_i' then we can remove these points of intersection by constructing a new orientable surface S_i' which spans k_i: cut holes in S_i' at p_1 and p_2 and connect the two added boundary curves by a small tube around the arc A. Continuing this process, we can remove either all $+$ intersection points, or all $-$ points of intersection of k_j with S_i' without introducing any new points of intersection. We have then an orientable surface S_i' which spans k_i and the number η_{ij} of points of intersection of k_j with S_i' is the absolute value of the linking number of k_i and k_j. Repeating this for each $j \neq i$, we ultimately ob-

Received by the editors April 13, 1966.

tain a surface S_i' which spans k_i and the number of points of intersection of S_i' with $\bigcup_{j\neq i} k_j$ is $\eta_i = \sum_{j,j\neq i} \eta_{ij}$. Hence there exists an S_i' of minimal genus with the above property; if S_i' is such a surface, let $S_i = S_i' - \bigcup_{j\neq i} k_j$. For convenience in this paper, we shall call S_i' a *pre-linking surface* and S_i a *linking surface* for k_i, and call the genus of S_i' the *linking genus* of k_i with respect to L.

Let U_i be a regular neighborhood (in the sense of Whitehead [3]) of the linking surface S_i for k_i such that Bd U_i contains k_i. (Bd U_i) $-k_i$ has two components S_i^1 and S_i^2. Lemmas 1 and 2 in §4 of [2] are now valid for S_i^1 and S_i^2.

LEMMA 1. *The inclusion map* $c\colon S_i^1 \to \text{Bd } U_i$ *induces a monomorphism* $c_*\colon \pi_1(S_i^1) \to \pi_1(\text{Bd } U_i)$.

LEMMA 2. *The inclusion map* $d\colon S_i^1 \to S^3 - (\text{Int}(U_i) \cup L)$ *induces a monomorphism* $d_*\colon \pi_1(S_i^1) \to \pi_1(S^3 - (\text{Int}(U_i) \cup L))$.

Some remarks on the proofs of these two lemmas are perhaps in order. In the proof of Lemma 1, k_i may be unknotted in which case S_i may be a disk. However, assuming L is not splittable, S_i^1 must be a disk with a number of points removed, so that $k_i \to S_i^1$ still induces a monomorphism. In the proof of Lemma 2, by assuming there is a closed curve α on S_i^1 such that $\alpha \cong 0$ in $S^3 - (\text{Int}(U_i) \cup L)$ but not on S_i^1, one either reduces the genus of S_i' or reduces the number η_i.

For each $i = 1, 2, \cdots, \mu$, let H_i be the normal subgroup of G consisting of all elements of G which are represented by loops in $S^3 - L$ which have linking number zero with k_i. Clearly $G/H_i = A_i$ so that the kernel of ϕ_i is H_i.

Using the linking surface S_i for k_i, we can now construct the covering space X_i of $S^3 - L$ corresponding to $H_i = \ker \phi_i$. This is done just as in [2], but we include a brief description of X_i for completeness.

Let $Y_0^i = S^3 - (\text{Int}(U_i) \cup L)$, and let $\{Y_j^i\}_{j=-\infty}^{\infty}$ be a countable collection of disjoint copies of Y_0^i. Each Y_j^i has two boundary components ${}_jS_i^1$ and ${}_jS_i^2$ which are homeomorphic in a natural way to ${}_0S_i^1 = S_i^1$ and ${}_0S_i^2 = S_i^2$, respectively. Also S_i^1 and S_i^2 are homeomorphic, so we construct X_i by attaching Y_j^i to Y_{j+1}^i, glueing ${}_jS_i^1$ to ${}_{j+1}S_i^2$ for each j, $-\infty < j < \infty$.

Clearly $\pi_1(Y_0^i) \subset H_i$, and since G/H_i is infinite cyclic, it follows that X_i is the covering space of $S^3 - L$ corresponding to H_i.

3. **Group structure.** The proof of Theorem 1 of [2] with $[G, G]$ replaced by H_i is valid. Thus we have:

THEOREM 3.1. *If* H_i *is finitely generated, it is free of rank* $2g_i + \eta_i$

where g_i is the linking genus of k_i with respect to L, and η_i is the sum of the absolute values of the linking numbers of k_i with k_j, $j \neq i$.

If H_i is not finitely generated, then either it is

(a) *a nontrivial free product with amalgamation on a free group F of rank $2g_i+\eta_i$,*

$$\cdots * A \underset{F}{*} A \underset{F}{*} A * \cdots,$$

or,

(b) *locally free and a direct limit of free groups of rank $2g_i+\eta_i$.*

The following lemma will be used in the sequel.

LEMMA 3.2. (i) $\bigcap_{i=1}^{k-1} H_i/\bigcap_{i=1}^{k} H_i = A_k$, $2 \leqq k \leqq \mu$.
(ii) $G/\bigcap_{i=1}^{k} H_i = A_1 \oplus A_2 \oplus \cdots \oplus A_k$, $1 \leqq k \leqq \mu$.

PROOF. We prove (i). (ii) is proved in a similar manner. For each $i=1, 2, \cdots, \mu$, $G \supset H_i \supset [G, G]$ since G/H_i is abelian. Thus

$$G/[G, G] = A_1 \oplus \cdots \oplus A_\mu \supset \bigcap_{i=1}^{m} H_i/[G, G] = A_{m+1} \oplus \cdots \oplus A_\mu$$

so that

$$\begin{aligned} \bigcap_{i=1}^{k-1} H_i \Big/ \bigcap_{i=1}^{k} H_i &= \left(\bigcap_{i=1}^{k-1} H_i/[G, G]\right) \Big/ \left(\bigcap_{i=1}^{k} H_i/[G, G]\right) \\ &= (A_k \oplus A_{k+1} \oplus \cdots \oplus A_\mu)/(A_{k+1} \oplus \cdots \oplus A_\mu) \\ &= A_k. \end{aligned}$$

We will now restrict our attention to the situation when $\mu = 2$ and the linking number of k_1 and k_2 is nonzero; i.e., $\eta_{12} = \eta_{21} = \eta_1 = \eta_2 \neq 0$. The covering space X_{12} of $S^3 - L$ corresponding to $H_1 \cap H_2$ will be constructed and, from this, the structure of $H_1 \cap H_2$ can be determined. Note that by Lemma 3.2, $H_1 \cap H_2 = \ker[G \to A_1 \oplus A_2] = [G, G]$.

Before constructing X_{12}, we need three lemmas. Let S_1', S_2', and S_1, S_2 be pre-linking and linking surfaces for k_1, k_2, respectively. We have already constructed the covering space X_1 of $S^3 - L$ corresponding to H_1. Let $p_1 \colon X_1 \to S^3 - L$ be the covering map.

LEMMA 3.3. *$p_1^{-1}(S_2)$ is a connected orientable 2-manifold in X_1.*

PROOF. Since $\eta_{12} \neq 0$, k_2 intersects S_1'. Intersections of S_1' and S_2' are either simple closed curves which lie on the interiors of S_1' and S_2', or arcs which run from k_1 to k_2 (no arc can run from k_i to k_i by the definition of S_1' and S_2'). It follows from this that all the components of $p_1^{-1}(k_2)$ lie on the same component of $p_1^{-1}(S_2)$. Hence, by the path lifting property, $p_1^{-1}(S_2)$ is connected.

LEMMA 3.4. *The inclusion* i: $p_1^{-1}(S_2)\to X_1$ *induces a monomorphism* i^*: $\pi_1(p_1^{-1}(S_2))\to\pi_1(X_1)$.

PROOF. Since $p_1^{-1}(S_2)$ is connected, it is a covering space over S_2 with covering map $q=p_1|p_1^{-1}(S_2)$.

The diagram

$$\begin{array}{ccc} \pi_1(p_1^{-1}(S_2)) & \xrightarrow{i^*} & \pi_1(X_1) \\ q^*\downarrow & & \downarrow p_1^* \\ \pi_1(S_2) & \xrightarrow{j^*} & G \end{array}$$

is commutative, j^* is induced by inclusion. q^*, j^*, and p_1^* are all monomorphisms, and hence so is i^*.

LEMMA 3.5. $X_1-p_1^{-1}(S_2)$ *is connected, and*

$$h^*(\pi_1(X_1 - p_1^{-1}(S_2)) \subset (p_1^*)^{-1}(H_1 \cap H_2) \subset \pi_1(X_1) \approx H_1,$$

where h^* *is induced by the inclusion map* h: $X_1 - p_1^{-1}(S_2)\to X_1$.

PROOF. $X_1-p_1^{-1}(S_2)$ is connected since $p_1^{-1}(S_2)$ is a connected 2-manifold with boundary $p_1^{-1}(k_2)$.

p_1h maps $X_1-p_1^{-1}(S_2)$ onto $S^3-(L\cup S_2)$, so $p_1^*h^*(\pi_1(X_1-p_1^{-1}(S_2)))\subset H_2$.

Now, by Lemma 3.2,

$$\pi_1(X_1)/(p_1^*)^{-1}(H_1 \cap H_2) \approx H_1/(H_1 \cap H_2) = A_2,$$

so we can construct the infinite cyclic covering space X_{12} over X_1 corresponding to $(p_1^*)^{-1}(H_1\cap H_2)$ with covering map p'_{12}. This is done just as X_1 was constructed, with $p_1^{-1}(S_2)$ replacing S_1. Thus X_{12} is the covering space of S^3-L corresponding to $H_1\cap H_2=[G, G]$ with covering map $p_{12}=p_1p'_{12}$.

Note that by the method of construction of X_1, $\pi_1(p_1^{-1}(S_2))$ is either trivial, or free and infinitely generated; so by Lemma 3.4 and the proof of Theorem 1 of [2], we have:

THEOREM 3.6. *If G is not $Z\oplus Z$, $[G, G]$ is infinitely generated and has one of the following forms*:

(a) *free*;

(b) *an infinite free product of isomorphic groups*;

(c) *an infinite free product of isomorphic groups with amalgamation* *on an infinitely generated free group;*

(d) *locally free and a direct limit of infinitely generated free groups.*

When $\mu>2$ and $\eta_{ij}\neq 0$ $(i\neq j)$ for each i, j, $1\leqq i\leqq\mu$, $1\leqq j\leqq\mu$, the above method can be used to construct the covering space of S^3-L corresponding to $\ker[G\to A_{i_1}\oplus\cdots\oplus A_{i_m}]$ so that Theorem 3.6 is valid with $[G, G]$ replaced by this subgroup. For, by reordering the components of L, we can assume that the homomorphism in question is $G\to A_1\oplus A_2\oplus\cdots\oplus A_m$, so that by Lemma 3.2, the kernel of the homomorphism is $\bigcap_{i=1}^m H_i$. Thus, we can construct a finite number of covering spaces $X_{12\cdots m}, X_{12\cdots(m-1)},\ldots,X_1$ of S^3-L such that $X_{12\cdots K}$ corresponds to $\bigcap_{i=1}^k H_i$. If $p_{12\cdots k}\colon X_{12\cdots k}\to S^3-L$ is the covering map, $X_{12\cdots(k+1)}$ is the infinite cyclic covering space of $X_{12\cdots k}$ corresponding to $(p^*_{12\cdots k})^{-1}(\bigcap_{i=1}^{k+1} H_i)$. $X_{12\cdots(k+1)}$, is of course constructed as X_{12} was constructed above, replacing $p_1^{-1}(S_2)$ by $p_{12\cdots k}^{-1}(S_{k+1})$. The requirement that $\eta_{ij}\neq 0$ insures that $p_{12\cdots k}^{-1}(S_{k+1})$ will be connected for each $k=1, 2, \cdots, \mu-1$. This allows the use of the Van Kampen theorem, in the proof of Theorem 3.6. However, should $\eta_{ij}=0$ for some i and j, the covering spaces can still be constructed even though $p_{12\cdots k}^{-1}(S_{k+1})$ will not necessarily be connected.

The following theorem gives a sufficient condition for $[G, G]$ to be a free product.

THEOREM 3.7. *If there is a component k_2 of L such that $\eta_2=1$ and the linking genus g_2 of k_2 is zero, and if $G\neq Z\oplus Z$, then $[G, G]$ is a free product.*

PROOF. Since $\eta_2=1$ and $g_2=0$, the linking surface S_2 of k_2 is a "half open" annulus. Let k_i be the component of L such that $\eta_{i2}=1$ and let S_i be its linking surface. If we assume $i=1$, we can construct the covering spaces X_1 and X_{12}. Since $\eta_{12}\neq 0$, $p_1^{-1}(S_2)$ is connected and hence is a covering space over S_2. Now $p_1^*\pi_1(p_1^{-1}(S_2))\subset H_1\cap\pi_1(S_2)$, and since $\eta_{12}=1$, $\pi_1(S_2)\cap H_1=0$. Hence $\pi_1(p_1^{-1}(S_2))$ is trivial. Since $H_1\cap H_2$ is a free product with amalgamation on $\pi_1(p_1^{-1}(S_2))$, $H_1\cap H_2$ is just a free product. $H_1\cap H_2\supset[G, G]$ and so by the Kurosh Subgroup Theorem [1], $[G, G]$ is a free product.

REFERENCES

1. A. G. Kurosh, *Theory of groups*, Vol. II, Chelsea, New York, 1960.

2. L. Neuwirth, *The algebraic determination of the genus of knots*, Amer. J. Math. **82** (1960), 791–798.

3. J. H. C. Whitehead, *Simplicial spaces, nuclei, and m-groups*, Proc. London Math. Soc. **45** (1939), 243–327.

THE UNIVERSITY OF GEORGIA