

AN INEQUALITY FOR PROBABILITIES

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1. The main result and applications. Given a probability measure space $(\Omega, \mathfrak{F}, P)$, let $A_k \in \mathfrak{F}$, $k=1, \dots, N$. The main result is given in the following theorem.

THEOREM 1.1.

$$(1.1) \quad P\left(\bigcup_{k=1}^N A_k\right) \geq \frac{\theta \mathfrak{B}^2}{2\alpha + (2 - \theta)\mathfrak{B}} + \frac{(1 - \theta)\mathfrak{B}^2}{2\alpha + (1 - \theta)\mathfrak{B}}$$

where $\mathfrak{B} = \sum_{k=1}^N P(A_k)$, $\alpha = \sum_{k=1}^N \sum_{l=1}^{k-1} P(A_k \cap A_l)$ and $\theta = 2\alpha/\mathfrak{B} - [2\alpha/\mathfrak{B}]$, $0 \leq \theta < 1$.

The proof of Theorem 1.1 is given in §2.

COROLLARY 1. A necessary condition for $P(\bigcup_{k=1}^N A_k) < 1$ is that $\mathfrak{B} < (1 + (1 + 8\alpha)^{1/2})/2$.

PROOF. It is easy to verify that if the right-hand side of inequality (1.1) is regarded as a function of θ , then the minimum occurs for $\theta = 0$. Hence

$$P\left(\bigcup_{k=1}^N A_k\right) \geq \frac{\mathfrak{B}^2}{2\alpha + \mathfrak{B}}$$

Therefore if $1 > P(\bigcup_{k=1}^N A_k)$, then $1 > \mathfrak{B}^2/(2\alpha + \mathfrak{B})$, that is, $\mathfrak{B}^2 - \mathfrak{B} - 2\alpha < 0$. Hence it is necessary that

$$\mathfrak{B} < (1 + (1 + 8\alpha)^{1/2})/2.$$

The next application is an elementary proof of the Erdős-Rényi form of the Borel-Cantelli lemma [1, p. 326].

COROLLARY 2. If $A_k \in \mathfrak{F}$, $k=1, 2, 3, \dots$, with $\sum_{k=1}^{\infty} P(A_k) = +\infty$, then $P(\bigcup_{k=1}^{\infty} A_k) \geq 1/c$ where

$$c = \liminf_{n \rightarrow \infty} \frac{\sum_{k=1}^n \sum_{l=1}^n P(A_k \cap A_l)}{\left(\sum_{k=1}^n P(A_k)\right)^2}.$$

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PROOF. By Theorem 1.1

$$P\left(\bigcup_{k=N}^M A_k\right) \geq \frac{\theta}{\frac{2\alpha_{NM}}{\mathfrak{B}_{NM}^2} + \frac{(2-\theta)}{\mathfrak{B}_{NM}}} + \frac{(1-\theta)}{\frac{2\alpha_{NM}}{\mathfrak{B}_{NM}^2} + \frac{(1-\theta)}{\mathfrak{B}_{NM}}}$$

where

$$\begin{aligned} 2\alpha_{NM} &= 2 \sum_{k=N}^M \sum_{l=N}^{k-1} P(A_k \cap A_l) \\ &= \sum_{k=N}^M \sum_{l=N}^M P(A_k \cap A_l) - \mathfrak{B}_{NM} \end{aligned}$$

and

$$\mathfrak{B}_{NM} = \sum_{k=N}^M P(A_k).$$

Hence since $\lim_{M \rightarrow \infty} \mathfrak{B}_{NM} = +\infty$,

$$\lim_{m \rightarrow \infty} P\left(\bigcup_{k=N}^M A_k\right) \geq \frac{1}{c_N}$$

where

$$\begin{aligned} c_N &= \liminf_{M \rightarrow \infty} \frac{\sum_{k=N}^M \sum_{l=N}^M P(A_k \cap A_l)}{\mathfrak{B}_{NM}^2} \\ &= c. \end{aligned}$$

Corollary 2 can also be proved by the following inequality due to Chung and Erdős [2],

$$(1.2) \quad P\left(\bigcup_{k=1}^N A_k\right) \geq \frac{\mathfrak{B}^2 - \mathfrak{B}}{2\alpha}.$$

(1.2) is easily proved by using Schwarz's inequality applied to indicator functions of sets.

However the inequality of Theorem 1.1 is stronger than (1.2) unless $(\mathfrak{B}^2 - \mathfrak{B})/2\alpha = 1$.

In fact it is easy to verify that

$$\frac{\mathfrak{B}^2}{2\alpha + \mathfrak{B}} > \frac{\mathfrak{B}^2 - \mathfrak{B}}{2\alpha} \quad \text{if } 0 \leq \frac{\mathfrak{B}^2}{2\alpha + \mathfrak{B}} < 1$$

and

$$\frac{\mathfrak{B}^2}{2\alpha + \mathfrak{B}} = \frac{\mathfrak{B}^2 - \mathfrak{B}}{2\alpha} \quad \text{if} \quad \frac{\mathfrak{B}^2}{2\alpha + \mathfrak{B}} = 1.$$

It will be shown elsewhere that the lower bound of Theorem 1.1 can be attained so that the inequality cannot be improved.

2. Proof of Theorem 1.1. Given $\{A_1, \dots, A_N\}$ we define B_r , $r=1, \dots, N$, to be the set of points in $\bigcup_{k=1}^N A_k$ which belong to exactly r of the sets $\{A_1, \dots, A_N\}$ and let $a_r = P(B_r)$.

Then we have

$$(2.1) \quad a_r \geq 0, \quad r = 1, \dots, N,$$

$$(2.2) \quad \sum_{k=1}^N k a_k = \sum_{k=1}^N P(A_k) = \mathfrak{B},$$

and

$$(2.3) \quad \sum_{k=2}^N \frac{k(k-1)}{2} a_k = \sum_{k=1}^N \sum_{l=1}^{k-1} P(A_k \cap A_l) = \alpha.$$

We proceed by finding the minimum of the linear expression

$$(2.4) \quad V = P\left(\bigcup_{k=1}^N A_k\right) = \sum_{k=1}^N a_k$$

subject to the constraints (2.1), (2.2) and (2.3). Substituting the expression for a_{r-1} which is obtained by solving (2.2) for a_{r-1} into (2.3), and dividing by r to make the coefficient of a_r equal to one we obtain

$$(2.5) \quad \begin{aligned} \frac{-(r-2)}{r} a_1 + \sum_{k=2}^N a_k \left\{ \frac{k(k-1)}{r} - \frac{k(r-2)}{r} \right\} \\ = \frac{2\alpha}{r} - \frac{(r-2)}{r} \mathfrak{B}. \end{aligned}$$

Substituting the expression for a_r which is obtained by solving (2.3) for a_r into (2.2), we obtain

$$(2.6) \quad a_1 + \sum_{k=2}^N a_k \left\{ k - \frac{k(k-1)}{r-1} \right\} = \mathfrak{B} - \frac{2\alpha}{r-1}.$$

Now solving (2.5) for a_r and (2.6) for a_{r-1} and substituting the resulting expressions into (2.4) we obtain

$$(2.7) \quad V - \frac{2\mathfrak{B}}{r} + \frac{2\alpha}{r(r-1)} = a_1 \frac{(r-2)}{r} + \sum_{k=2}^N \frac{(r-k)(r-k-1)}{r(r-1)} a_k.$$

We henceforth assume that in equations (2.5), (2.6) and (2.7),

$$(2.8) \quad \begin{aligned} r &= 2 + [2\alpha/\mathfrak{B}] \quad \text{if } 2\alpha/\mathfrak{B} \neq (N-1) \\ &= N \quad \text{if } 2\alpha/\mathfrak{B} = N-1. \end{aligned}$$

Since $2\alpha \leq (N-1)\mathfrak{B}$ it follows that $r \leq N$.

It then follows that

$$(2.9) \quad (r-2)/r \geq 0$$

and

$$(2.10) \quad (r-k)(r-k-1)/r(r-1) \geq 0, \quad k = 2, \dots, N.$$

Since (2.9) and (2.10) imply that the coefficients of the a_r in the right-hand side of equation (2.7) are nonnegative, the minimal value of $V - 2\mathfrak{B}/r + 2\alpha/r(r-1)$ is zero.

But if we set

$$(2.11) \quad \begin{aligned} a_{r-1} &= \mathfrak{B} - 2\alpha/(r-1), \\ a_r &= 2\alpha/r - (r-2)\mathfrak{B}/r, \\ a_k &= 0, \quad k \neq r-1, r, \end{aligned}$$

then equations (2.5) and (2.6) are satisfied and $V - 2\mathfrak{B}/r + 2\alpha/r(r-1) = 0$. Moreover (2.8) implies that $a_{r-1} \geq 0$ and $a_r \geq 0$ so that the set of a_n 's given by (2.11) minimizes the expression (2.4) and satisfies all the constraints.

Therefore the minimal value of V is

$$\begin{aligned} V &= \frac{2\mathfrak{B}}{r} - \frac{2\alpha}{r(r-1)} \\ &= \frac{2\mathfrak{B}^2}{2\alpha + (2-\theta)\mathfrak{B}} - \frac{2\alpha\mathfrak{B}^2}{(2\alpha + (2-\theta)\mathfrak{B})(2\alpha + (1-\theta)\mathfrak{B})} \end{aligned}$$

where $\theta = 2\alpha/\mathfrak{B} - [2\alpha/\mathfrak{B}]$, $0 \leq \theta < 1$. But then

$$V = \frac{\theta\mathfrak{B}^2}{2\alpha + (2-\theta)\mathfrak{B}} + \frac{(1-\theta)\mathfrak{B}^2}{2\alpha + (1-\theta)\mathfrak{B}}.$$

REFERENCES

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