ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF AN nTH-ORDER NONLINEAR EQUATION[1]

PHIL LOCKE

Introduction. We consider the equation

$$x^{(n)} + \sum_{i=0}^{n-1} \sum_{j=1}^{m_i} a_{ij}(t)\left[x^{(i)}\right]^{r_{ij}} = 0 \tag{1}$$

and the more general equation

$$\begin{aligned} x^{(n)} + b_{n-1}(t)x^{(n-1)} + \cdots + b_1(t)x' + b_0(t)x \\ + \sum_{i=0}^{n-1} \sum_{j=1}^{m_i} a_{ij}(t)\left[x^{(i)}\right]^{r_{ij}} = 0. \end{aligned} \tag{2}$$

We tacitly assume throughout that all coefficient functions are continuous, that $n \geqq 2$, and that each r_{ij} is of the form p_{ij}/q_{ij} where p_{ij} is a positive integer and q_{ij} is an odd positive integer. Roughly speaking, we assume that the above equations are "almost linear" in the sense that the nonlinear terms go to zero as t becomes large. It then follows that solutions of (1) and (2) "behave like" solutions of the corresponding linear equations

$$x^{(n)} = 0, \tag{1L}$$

$$x^{(n)} + b_{n-1}(t)x^{(n-1)} + \cdots + b_1(t)x' + b_0(t)x = 0 \tag{2L}$$

as t becomes large.

Theorems 1, 3, 4, and 5 are generalizations of results obtained by Waltman [1] for the equations

(1*) $x'' + a(t)x^{2n+1} = 0$, $n \geqq 1$ and

(2*) $x'' - f(t)x + q(t)x^{2n+1} = 0$, $n \geqq 1$.

Theorem 2 is of a converse nature; it generalizes a result obtained by Moore and Nehari [2, pp. 36–37] for the equation (1*) when $a(t) \geqq 0$.

Preliminary lemma. Waltman [1] bases his proofs on a powerful theorem by B. Viswanatham. We shall instead use the following result, which is an easy generalization of Gronwall's inequality. We give the proof here for the sake of completeness.

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[1] The results of this paper have been accepted as part of my doctoral dissertation by the University of New Hampshire.

LEMMA. *Assume* $y(t) \leqq \delta + \int_{t_1}^{t} p(\tau) y^r(\tau) d\tau$ *for all* $t \geqq t_1 \geqq t_0$, *where* $\delta > 0$, $r \geqq 1$, *and* $y(t)$ *and* $p(t)$ *are continuous and nonnegative for* $t \geqq t_1$, $t \geqq t_0$, *respectively.*

Assume $\int_{t_0}^{\infty} p(t) dt < \infty$. *If* $r > 1$ *we require, in addition, that* (t_0, δ) *satisfy*

$$\delta < \left[(r-1) \int_{t_0}^{\infty} p(t) dt \right]^{-1/(r-1)}. \tag{3}$$

Conclusion: $y(t)$ *is bounded on* $[t_1, \infty)$. (*The bound is independent of* t_1.)

PROOF. Put $\theta(t) = \delta + \int_{t_1}^{t} p(\tau) y^r(\tau) d\tau$. Hence $\theta'(t) = p(t) y^r(t) \leqq p(t) \theta^r(t)$; thus

$$\int_{t_1}^{t} \theta^{-r}(\tau) \theta'(\tau) d\tau \leqq \int_{t_1}^{t} p(\tau) d\tau \leqq \int_{t_0}^{\infty} p(t) dt.$$

If $r = 1$ we have

$$\theta(t) \leqq \delta \exp\left[\int_{t_0}^{\infty} p(t) dt \right], \qquad t \geqq t_1.$$

If $r > 1$ we have [using (3)]

$$\theta(t) \leqq \left[\delta^{-(r-1)} - (r-1) \int_{t_0}^{\infty} p(t) dt \right]^{-1/(r-1)}, \qquad t \geqq t_1.$$

Since $y(t) \leqq \theta(t)$ on $[t_1, \infty]$, the result follows.

We remark that the lemma is also true when $0 < r < 1$, but this result is not needed.

Results.

THEOREM 1. *If* $\int^{\infty} \sum_{i=0}^{n-1} \sum_{j=1}^{m_i} |a_{ij}(t)| t^{(n-1-i) r_{ij}} dt < \infty$ *then equation* (1) *has solutions* $x(t)$ *such that*

$$\lim_{t \to \infty} x^{(n-1)}(t) = \alpha > 0 \quad (\alpha \textit{ finite}).$$

PROOF. Let r denote the maximum of 1 and all the r_{ij}. Let $x(t)$ be a solution of (1). If $r > 1$ we require that $x(t)$ satisfy the initial value inequality

$$\begin{aligned} &\max\left[1, \sum_{p=0}^{n-1} \sum_{k=0}^{n-1-p} \frac{|x^{(p+k)}(t_0)|}{k! t_0^{n-1-p-k}} \right] \\ &< \left[(r-1) \sum_{p=0}^{n-1} \frac{1}{p!} \int_{t_0}^{\infty} \sum_{i=0}^{n-1} \sum_{j=1}^{m_i} |a_{ij}(t)| t^{(n-1-i) r_{ij}} dt \right]^{-1/(r-1)} \end{aligned} \tag{4}$$

for some $t_0 > 0$.

Applying Taylor's formula with integral remainder, we have

$$x^{(p)}(t) = \sum_{k=0}^{n-1-p} \frac{x^{(p+k)}(t_0)}{k!}(t-t_0)^k + \frac{1}{(n-1-p)!}\int_{t_0}^{t}(t-\tau)^{n-1-p}x^{(n)}(\tau)d\tau,$$

$$p = 0, 1, \cdots, n-1, \qquad t \geqq t_0 > 0.$$

Hence

$$\frac{x^{(p)}(t)}{t^{n-1-p}} = \sum_{k=0}^{n-1-p} \frac{x^{(p+k)}(t_0)}{k!t^{n-1-p-k}}\left(\frac{t-t_0}{t}\right)^k - \frac{1}{(n-1-p!)} \cdot \int_{t_0}^{t}\left(\frac{t-\tau}{t}\right)^{n-1-p}\sum_{i=0}^{n-1}\sum_{j=1}^{m_i} a_{ij}(\tau)\tau^{(n-1-i)r_{ij}}\left[\frac{x^{(i)}(\tau)}{\tau^{n-1-i}}\right]^{r_{ij}} d\tau,$$

from which we obtain the inequalities

$$\frac{|x^{(p)}(t)|}{t^{n-1-p}} \leqq \sum_{k=0}^{n-1-p}\frac{|x^{(p+k)}(t_0)|}{k!t_0^{n-1-p-k}} + \frac{1}{(n-1-p)!} \cdot \int_{t_0}^{t}\sum_{i=0}^{n-1}\sum_{j=1}^{m_i}|a_{ij}(\tau)|\tau^{(n-1-i)r_{ij}}\left\{\max\left[1, \sum_{k=0}^{n-1}\frac{|x^{(k)}(\tau)|}{\tau^{n-1-k}}\right]\right\}^r d\tau.$$

Summing over p we obtain the inequality

$$\max\left[1, \sum_{p=0}^{n-1}\frac{|x^{(p)}(t)|}{t^{n-1-p}}\right]$$

$$\text{(5)} \qquad \leqq \max\left[1, \sum_{p=0}^{n-1}\sum_{k=0}^{n-1-p}\frac{|x^{(p+k)}(t_0)|}{k!t_0^{n-1-p-k}}\right] + \sum_{p=0}^{n-1}\frac{1}{p!} \cdot \int_{t_0}^{t}\sum_{i=0}^{n-1}\sum_{j=1}^{m_i}|a_{ij}(\tau)|\tau^{(n-1-i)r_{ij}}\left\{\max\left[1, \sum_{k=0}^{n-1}\frac{|x^{(k)}(\tau)|}{\tau^{n-1-k}}\right]\right\}^r d\tau.$$

It follows from the lemma that $\max\,[1, \sum_{p=0}^{n-1}|x^{(p)}(t)|/t^{n-1-p}]$ is bounded on $[t_0, \infty)$.

It now follows from

$$\text{(6)} \qquad x^{(n-1)}(t) = x^{(n-1)}(t_0) - \int_{t_0}^{t}\sum_{i=0}^{n-1}\sum_{j=1}^{m_i} a_{ij}(\tau)\tau^{(n-1-i)r_{ij}}\left[\frac{x^{(i)}(\tau)}{\tau^{n-1-i}}\right]^{r_{ij}} d\tau$$

that $\lim_{t\to\infty}x^{(n-1)}(t)$ exists (finite).

It remains to be shown that there always exist solutions $x(t)$ such that this limit is greater than zero. This is done by selecting appropriate initial conditions:

Choose $t_0>0$, $\delta\geqq 1$. If $r>1$ we require in addition that (t_0, δ) satisfy

$$\delta < \left[(r-1) \sum_{p=0}^{n-1} \frac{1}{p!} \int_{t_0}^{\infty} \sum_{i=0}^{n-1} \sum_{j=1}^{m_i} \left| a_{ij}(t) \right| t^{(n-1-i) r_{ij}} dt \right]^{-1/(r-1)}.$$

By the lemma there exists a bound M such that, if $y(t)$ is any nonnegative continuous function satisfying

$$y(t) \leqq \delta + \sum_{p=0}^{n-1} \frac{1}{p!} \int_{t_1}^{t} \sum_{i=0}^{n-1} \sum_{j=1}^{m_i} \left| a_{ij}(\tau) \right| \tau^{(n-1-i) r_{ij}} y^r(\tau) d\tau$$

for all $t \geqq t_1 \geqq t_0$, then $y(t) \leqq M$ on $[t_1, \infty)$.

Choose t_1 so large that

$$\int_{t_1}^{\infty} \sum_{i=0}^{n-1} \sum_{j=1}^{m_i} \left| a_{ij}(t) \right| t^{(n-1-i) r_{ij}} dt < \frac{\delta}{M^r \sum_{p=0}^{n-1} 1/p!}.$$

Let $x(t)$ be the solution of (1) satisfying the initial conditions $x(t_1) = x'(t_1) = \cdots = x^{(n-2)}(t_1) = 0$, $x^{(n-1)}(t_1) = \delta / \sum_{p=0}^{n-1} (1/p!)$. Proceeding as before, we obtain the inequality (5) with t_0 replaced by t_1. Since

$$\max \left[1, \sum_{p=0}^{n-1} \sum_{k=0}^{n-1-p} \frac{\left| x^{(p+k)}(t_1) \right|}{k! t^{n-1-p-k}} \right] = \delta,$$

it follows by the lemma that $\max[1, \sum_{p=0}^{n-1} |x^{(p)}(t)| / t^{n-1-p}]$ is bounded by M on $[t_1, \infty)$. Hence

$$\begin{aligned} \lim_{t\to\infty} x^{(n-1)}(t) &= x^{(n-1)}(t_1) \\ &\quad - \int_{t_1}^{\infty} \sum_{i=0}^{n-1} \sum_{j=1}^{m_i} a_{ij}(\tau) \tau^{(n-1-i) r_{ij}} \left[\frac{x^{(i)}(\tau)}{\tau^{n-1-i}} \right]^{r_{ij}} d\tau \\ &\geqq \frac{\delta}{\sum_{p=0}^{n-1} 1/p!} - M^r \int_{t_1}^{\infty} \sum_{i=0}^{n-1} \sum_{j=1}^{m_i} \left| a_{ij}(t) \right| t^{(n-1-i) r_{ij}} dt \\ &> 0. \end{aligned}$$

This completes the proof.

REMARK. Let $x(t)$ be any solution of (1). If $r>1$, require that $x(t)$ satisfy (4) for some $t_0>0$. Then, by the preceding proof, $\lim_{t\to\infty} x^{(n-1)}(t)$ exists (finite).

The next theorem is a weak converse of Theorem 1.

THEOREM 2. *Consider the equation* (1). *Assume, in addition to the usual hypotheses, that all the* $a_{ij}(t)$ *are eventually positive. Assume that*

(1) *has a solution* $x(t)$ *such that* $\lim_{t\to\infty} x^{(n-1)}(t)=\alpha>0$ (α *finite*). *Then*

$$\int^{\infty} \sum_{i=0}^{n-1} \sum_{j=1}^{m_i} a_{ij}(t) t^{(n-1-i) r_{ij}} dt < \infty.$$

Proof. By L'Hospital's rule

$$\lim_{t\to\infty} \frac{x^{(i)}(t)}{t^{n-1-i}} = \frac{\alpha}{(n-1-i)!}, \qquad i = 0, 1, \cdots, n-2.$$

Choose $t_0>0$ so large that $x^{(i)}(t)>0$ and $a_{ij}(t)>0$ for $t\geqq t_0$, all i, j. From (6) it follows that the improper integral

$$\int_{t_0}^{\infty} \sum_{i=0}^{n-1} \sum_{j=1}^{m_i} a_{ij}(t) t^{(n-1-i) r_{ij}} \left[\frac{x^{(i)}(t)}{t^{n-1-i}}\right]^{r_{ij}} dt$$

is convergent. Since all terms in the integrand are positive and since

$$\lim_{t\to\infty} \frac{x^{(i)}(t)}{t^{n-1-i}} = \frac{\alpha}{(n-1-i)!}, \qquad i = 0, 1, \cdots, n-1,$$

the result follows.

We now turn our attention to the more general equation (2).

Theorem 3. *Consider equation* (2). *Let* $x_1(t), x_2(t), \cdots, x_n(t)$ *be linearly independent solutions of the associated linear equation* (2L). *Let* $W(t)$ *denote their Wronskian.*

Let $X_i(t)=\max_{1\leq k\leq n} |x_k^{(i)}(t)|$, $i=0, 1, \cdots, n-1$.

Let $W_k(t)=$ *Wronskian* $[x_1(t), \cdots, x_{k-1}(t), x_{k+1}(t), \cdots, x_n(t)]$, $k=1, 2, \cdots, n$;

$$Z(t) = \max_{1\leqq k\leqq n} |W_k(t)|.$$

Assume

$$\int^{\infty} \frac{Z(t)}{|W(t)|} \sum_{i=0}^{n-1} \sum_{j=1}^{m_i} |a_{ij}(t)| X_i^{r_{ij}}(t) dt < \infty.$$

Conclusion: *Equation* (2) *has* (*nontrivial*) *solutions* $x(t)$ *such that*

$$x^{(p)}(t) = \sum_{k=1}^{n} A_k(t) x_k^{(p)}(t), \qquad p = 0, 1, \cdots, n-1 \tag{7}$$

where the functions $A_k(t)$ *are such that* $\lim_{t\to\infty} A_k(t)=a_k$ (*finite*), $k=1, 2, \cdots, n$.

Proof. Let $x(t)$ be a solution of (2). Let functions $A_k(t)$ be (uniquely) defined in terms of $x(t)$, $x'(t)$, $\cdots$, $x^{(n-1)}(t)$ by equations (7). Let r be defined as before. If $r>1$ we require that the initial value inequality

$$\max\left[1, \sum_{p=1}^{n} | A_p(t_0) | \right] < \left[n(r-1)\int_{t_0}^{\infty} \frac{Z(t)}{| W(t) |} \sum_{i=0}^{n-1}\sum_{j=1}^{m_i} | a_{ij}(t) | X_i^{r_{ij}}(t)dt\right]^{-1/(r-1)} \tag{8}$$

be satisfied for some t_0.

From equations (7) and the fact that $x(t)$ is a solution of (2) we obtain the equations

$$\begin{aligned} &\sum_{k=1}^{n} A_k'(t)x_k^{(p)}(t) = 0, \qquad p = 0, 1, \cdots, n-2, \\ &\sum_{k=1}^{n} A_k'(t)x_k^{(n-1)}(t) = -\sum_{i=0}^{n-1}\sum_{j=1}^{m_i} a_{ij}(t)\left[\sum_{k=1}^{n} A_k(t)x_k^{(i)}(t)\right]^{r_{ij}}. \end{aligned} \tag{9}$$

Solving this system for $A_1'(t)$, $A_2'(t)$, $\cdots$, $A_n'(t)$ and integrating from t_0 to t, we have

$$\begin{aligned} A_p(t) = A_p(t_0) + (-1)^{n+p+1}\int_{t_0}^{t} \frac{W_p(\tau)}{W(\tau)}\sum_{i=0}^{n-1}\sum_{j=1}^{m_i} a_{ij}(\tau) \\ \cdot\left[\sum_{k=1}^{n} A_k(\tau)x_k^{(i)}(\tau)\right]^{r_{ij}} d\tau, \qquad p = 1, 2, \cdots, n, \end{aligned} \tag{10}$$

from which we obtain the inequality

$$\begin{aligned} \max\left[1, \sum_{p=1}^{n} | A_p(t) | \right] \leqq \max\left[1, \sum_{p=1}^{n} | A_p(t_0) | \right] \\ + n\int_{t_0}^{t} \frac{Z(\tau)}{| W(\tau) |}\sum_{i=0}^{n-1}\sum_{j=1}^{m_i} | a_{ij}(\tau) | X_i^{r_{ij}}(\tau) \\ \cdot\left\{\max\left[1, \sum_{k=1}^{n} | A_k(\tau) | \right]\right\}^{r} d\tau. \end{aligned}$$

It now follows by the lemma that $\max[1, \sum_{p=1}^{n}| A_p(t)|]$ is bounded on $[t_0, \infty)$. Hence the integrals in equations (10) are (absolutely) convergent as $t\to\infty$, so $\lim_{t\to\infty}A_p(t)$ exists (finite), $p=1, 2, \cdots, n$. This completes the proof.

THEOREM 4. *Hypotheses: The same as those of Theorem 3. In addition, we also assume the following:*

If $x_k^{(p)}(t)$ *is unbounded as* $t \to \infty$ *then there exists a (positive) non-decreasing continuous function* $x_{kp}^*(t)$ *such that* $|x_k^{(p)}(t)| \leqq x_{kp}^*(t)$ *for large* t *and*

$$\int^{\infty} \frac{x_{kp}^*(t) \, | W_k(t) |}{| W(t) |} \sum_{i=0}^{n-1} \sum_{j=1}^{m_i} | a_{ij}(t) | \, X_i^{r_{ij}}(t) dt < \infty .$$

Conclusion: Equation (2) *has (nontrivial) solutions* $x(t)$ *such that*

$$x^{(p)}(t) = \sum_{k=1}^{n} a_k x_k^{(p)}(t) + \epsilon^{(p)}(t), \qquad p = 0, 1, \cdots, n-1,$$

where the a_k *are constant and* $\lim_{t\to\infty} \epsilon^{(p)}(t) = 0$.

Proof. We use the notation and results of Theorem 3. We first establish that

$$\lim_{t \to \infty} | A_k(t) - a_k | \, | x_k^{(p)}(t) | = 0.$$

This obviously holds when $x_k^{(p)}(t)$ is bounded, so assume that $x_k^{(p)}(t)$ is unbounded. Solving for $A_k'(t)$ in (9) and then integrating from t to ∞ we have

$$a_k = A_k(t) + (-1)^{n+k+1} \int_t^{\infty} \frac{W_k(\tau)}{W(\tau)} \sum_{i=0}^{n-1} \sum_{j=1}^{m_i} a_{ij}(\tau) \left[\sum_{k=1}^{n} A_k(\tau) x_k^{(i)}(\tau) \right]^{r_{ij}} d\tau .$$

Hence, for large t,

$$| A_k(t) - a_k | \, | x_k^{(p)}(t) | \leqq \int_t^{\infty} \frac{x_{kp}^*(\tau) \, | W_k(\tau) |}{| W(\tau) |} \sum_{i=0}^{n-1} \sum_{j=1}^{m_i} | a_{ij}(\tau) | \, X_i^{r_{ij}}(\tau) \cdot \left\{ \max \left[1, \sum_{k=1}^{n} | A_k(\tau) | \right] \right\}^{r} d\tau .$$

Hence $\lim_{t\to\infty} |A_k(t) - a_k| \, |x_k^{(p)}(t)| = 0$. Thus

$$x^{(p)}(t) = \sum_{k=1}^{n} A_k(t) x_k^{(p)}(t) = \sum_{k=1}^{n} a_k x_k^{(p)}(t) + \epsilon_p(t)$$

where $\lim_{t\to\infty} \epsilon_p(t) = 0$. Putting $\epsilon_0(t) = \epsilon(t)$, we must have $\epsilon_p(t) = \epsilon^{(p)}(t)$ and the theorem is proved.

Remark. Let $x(t)$ be any solution of (2). If $r > 1$, require that $x(t)$ be such that (8) is satisfied for some t_0. Let all hypotheses of Theorem 3 (Theorem 4) be satisfied. Then, by the preceding proofs, $x(t)$ has the asymptotic behavior given in the conclusion of Theorem 3 (Theorem 4).

If we apply Theorem 4 to the equation (1) we obtain the following result

THEOREM 5. *If*

$$\int^{\infty} t^{n-1} \sum_{i=0}^{n-1} \sum_{j=1}^{m_i} | a_{ij}(t) | t^{(n-1-i)r_{ij}} dt < \infty$$

then (1) *has* (*nontrivial*) *solutions* $x(t)$ *such that*

$$x(t) = \sum_{k=1}^{n} a_k t^{k-1} + \epsilon(t)$$

where

$$\lim_{t\to\infty} \epsilon^{(p)}(t) = 0, \qquad p = 0, 1, \cdots, n-1.$$

PROOF. Using the notation of Theorems 3 and 4 we have $x_k(t) = t^{k-1}$, $k = 1, 2, \cdots, n$; $W(t) =$ constant. We can take $X_i(t) = (n-1)!t^{n-1-i}$ for $t \geqq 1$, $i = 0, 1, \cdots, n-1$.

$W_k(t)$ is of the form $c_k t^{n-k}$ and so we can take $Z(t) = ct^{n-1}$ for $n \geqq 1$. It is now easily verified that all hypotheses of Theorem 4 are satisfied and the theorem is proved.

REFERENCES

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UNIVERSITY OF NEW HAMPSHIRE