ORTHOGONAL SERIES AND PROBABILITY

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1. **Introduction.** Recently R. Gundy [1] gave a characterization of Haar-like orthonormal systems based on martingale theory and used this theory to establish the capabilities of these systems for representing measurable functions. In this paper we will use this characterization of these orthonormal systems in deriving results related to recent work in Haar series [2], [3] and [4].

2. **Definitions.** Let $(X, \mathcal{S}, \mu)$ be a probability space. [See [5] for probability theory concepts.] An orthonormal system $\{\phi_n\}$ on $(X, \mathcal{S}, \mu)$ is called an *H-system* if the following conditions are satisfied:

(1) Each ϕ_n assumes at most two nonzero values with positive probability,

(2) the σ-field generated by $\{\phi_n\}_{n=1}^N$, written $\sigma(\phi_1, \phi_2, \cdots, \phi_N)$, consists of exactly N atoms,

(3) $E(\phi_{n+1}|\phi_1, \cdots, \phi_n)=0$, $n\geqq 1$, that is, the conditional expectation of ϕ_{n+1} with respect to $\sigma(\phi_1, \cdots, \phi_n)$ is zero.

In [1] it is shown that this is equivalent to $\{\phi_n\}$ having the property, that for every f in $L^2(X)$,

$$E(f|\phi_1, \cdots, \phi_n) = \sum_{k=1}^{n} a_k\phi_k,$$

where $\{a_k\}$ are the Fourier coefficients of f with respect to $\{\phi_n\}$.

H-systems are also easily seen to be equivalent to the class of orthonormal systems introduced by Price [3] and include the classical Haar system when the probability space is Lebesgue measure on $[0, 1]$. [See [6] for the definition and some of the properties of the classical Haar system.]

3. **Nonnegative Dirichlet kernels.** If $\{\phi_n\}$ is an orthonormal system on $(X, \mathcal{S}, \mu)$, then the Dirichlet kernels are

$$D_n(x, y) = \sum_{i=1}^{n} \phi_i(x)\phi_i(y).$$

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For the classical Haar system these kernels are nonnegative and this fact leads to the proof that the expansion of a continuous function on [0, 1] in a Haar series converges uniformly to the function. The following theorem, due to Price [3], has a more direct proof using probabalistic notions.

THEOREM 1. *The Dirichlet kernels of an orthonormal system* $\{\phi_n\}$ *with* $\phi_1 \equiv 1$ *are nonnegative if and only if* $\{\phi_n\}$ *is an H-system.*

PROOF. If $\{\phi_n\}$ is an H-system, then

$$E(f \mid \phi_1, \cdots, \phi_n) = \sum_{k=1}^{n} a_k \phi_k = \int_X D_n(\cdot, t) f(t) dt$$

for every nonnegative f in $L^2(X, \mathcal{S}, \mu)$. Since the conditional expectation is a positive operator, the integral is nonnegative for every $f \geqq 0$ which implies $D_n(x, t) \geqq 0$.

To show that the Dirichlet kernels being nonnegative implies that $\{\phi_n\}$ is an H-system we will show by induction that the three conditions defining an H-system are satisfied.[2] For $n=1$ there is nothing to prove since $\phi_1 \equiv 1$. Suppose the conditions are true for $2, \cdots, n$. Let $A_1, A_2, \cdots, A_n$ be the atoms of $\sigma(\phi_1, \cdots, \phi_n)$. Then, for fixed s, $D_n(s, t)$ must be constant for t in each atom. Therefore, for $1 \leqq k \leqq n$ and χ_A denoting the characteristic function of A,

$$\begin{aligned} \chi_{A_k} &= E(\chi_{A_k} \mid \phi_1, \cdots, \phi_n) = \int_X \chi_{A_k}(t) D_n(\cdot, t) d\mu(t) \\ &= \mu(A_k) D_n(\cdot, t) \big|_{t \in A_k}. \end{aligned}$$

Then,

$$\begin{aligned} D_n(s, t) &= 1/\mu(A_k), \ (s, t) \in A_k \times A_k, \quad k = 1, 2, \cdots, n, \\ &= 0, \quad \text{otherwise}. \end{aligned} \tag{1}$$

Since $D_{n+1}(s, t) \geqq 0$, $\phi_{n+1}(s)\phi_{n+1}(t) \geqq -D_n(s, t)$. Now, ϕ_{n+1} is zero on all but one of the atoms $A_1, A_2, \cdots, A_n$ since, if not, $\phi_{n+1}(s)\phi_{n+1}(t) \geqq 0$ everywhere by (1) and this contradicts the fact that $\int \phi_{n+1} d\mu = 0$ because ϕ_{n+1} is orthogonal to ϕ_1. Therefore, condition (3) of the definition is satisfied.

If A_k is the atom of $\sigma(\phi_1, \cdots, \phi_n)$ on which ϕ_{n+1} is nonzero, set $P = \{x : \phi_{n+1} > 0\}$ and $Q = A_k - P$. Let c_1 be the essential supremum of ϕ_{n+1} and c_2 be the essential infimum of ϕ_{n+1}. The essential infimum of $\phi_{n+1}(s)\phi_{n+1}(t)$ for s in P and t in Q must be $c_1 c_2$. Since ϕ_{n+1} is subject

[2] An alternate proof using idempotent contractions has been communicated to the author by R. F. Gundy.

to the orthonormality conditions, the minimum value of c_1 occurs when $\phi_{n+1}(s) \equiv c_1$ a.e. on P and the maximum value of c_2 occurs when $\phi_{n+1}(t) \equiv c_2$ a.e. on Q. In this case we find $c_1c_2 = -1/\mu(A_k)$. Therefore, if ϕ_{n+1} is not constant on P and Q, the essential infimum of $\phi_{n+1}(s)\phi_{n+1}(t)$ is less than $-1/\mu(A_k)$ which contradicts $D_{n+1}(s, t) \geqq 0$. Consequently, we have conditions (1) and (2) of the definition of an H-system also satisfied.

4. **Convergence of *H*-series.** It follows from the martingale convergence theorem [5, p. 331] that the expansion of an integrable function in an H-series converges a.e., converging to the function if the H-system is complete. Clearly, any sequence $\{a_k\}$ such that $\sum a_k^2 < \infty$ is a coefficient sequence for which $\sum a_k\phi_k$ converges a.e. for any H-system $\{\phi_n\}$. We wish to examine H-series $\sum a_k\phi_k$ for which $\sum a_k^2 = \infty$.

An increasing sequence of atomic σ-fields is regular if there exists a $\delta > 0$ such that for atoms $E_n \in \sigma(n)$ and $E_{n+1} \in \sigma(n+1)$ with $E_{n+1} \subset E_n$, $\mu(E_{n+1})/\mu(E_n) > \delta$.

The following will prove useful.

Lemma 1 (Gundy). *Let $\{\phi_k\}$ be an H-system such that $\sigma(n) = \sigma(\phi_1, \cdots, \phi_n)$ is a regular sequence of σ-fields. Then the series $\sum a_k\phi_k$ converges a.e. on a set E if and only if $\sum (a_k\phi_k)^2$ converges a.e. on E.*

The proof (see Theorem 3.1 of [1]) uses several ideas from martingale theory.

Given x, let E_n be the atom of $\sigma(n)$ containing x. If $\phi_n(x) = 0$ for all n sufficiently large then $\sum a_k\phi_k(x)$ converges for all sequences $\{a_k\}$. Therefore, let E be the set of all x for which the sequence of positive integers, $n_k = n_k(x)$, such that $\phi_{n_k}(x) \neq 0$ is infinite, and let $\epsilon_1 = 1$ and $\epsilon_k = 1/\mu(E_{n_{k-1}})$ for $k \geqq 2$.

Theorem 2. *Let $\{\phi_k\}$ be an H-system satisfying the conditions of Lemma 1. Then $\sum a_k\phi_k$ converges a.e. on E if and only if $\sum a_{n_k}^2\epsilon_k$ converges for almost all x in E.*

Proof. By the properties of H-systems, ϕ_{n_k} is nonzero only on $E_{n_k\Delta 1}$ and must be constant on E_{n_k} and $E_{n_k\Delta 1} - E_{n_k}$. By the orthonormality of ϕ_n, we have that on E_{n_k},

$$\phi_{n_k}^2 = \mu(E_{n_{k-1}} - E_{n_k})/\mu(E_{n_{k-1}})\mu(E_{n_k}).$$

Since $\sigma(n)$ is regular, there exists a δ, $0 < \delta < 1$, such that

$$\delta/(1-\delta)\mu(E_{n_{k-1}}) \leqq \phi_{n_k}^2(x) \leqq (1-\delta)/\delta\mu(E_{n_{k-1}}).$$

The theorem then follows by Lemma 1.

For the remainder of this paper we will want to compare the behavior of H-systems with the classical Haar system and will assume that the measure space is the unit interval.

The following result, proved recently by Leindler [**4**], follows easily from Theorem 2.

COROLLARY. *If* $\sum a_k^2 = \infty$ *and* $a_{k+1}^2 \leqq a_k^2$, *then* $\sum a_k h_k$ *diverges a.e., where* $\{h_k\}$ *is the Haar orthonormal system.*

PROOF. If x is not a binary fraction, then

$$\sum a_{n_k}^2 \epsilon_k = \sum a_{n_k}^2 2^k,$$

where $2^k < n_k \leqq 2^{k+1}$. But this series diverges by Cauchy's condensation principle.

The preceding corollary does not hold for all H-systems, even if $\sigma(n)$ is a regular sequence of σ-fields.

THEOREM 3. *Let* μ *be a nonatomic measure on* $[0, 1]$.

(a) *If* $\{a_k\}$ *is a sequence such that* $\liminf |a_k| = 0$, *then there exists an* H-*system, with* $\sigma(\phi_1, \phi_2, \cdots)$ *nonatomic, such that* $\sum a_k \phi_k$ *converges a.e.*

(b) *If* $\{a_k\}$ *is a sequence such that* $\limsup |a_k| > 0$, *then there exists an* H-*system, with* $\sigma(\phi_1, \phi_2, \cdots)$ *nonatomic, such that* $\sum a_k \phi_k$ *diverges a.e.*

PROOF. Let $I(0, 0) = [0, 1]$ and let $I(k, n)$, $n \geqq 1$, $0 \leqq k \leqq 2^n - 1$, denote a partition of $I(0, 0)$ into disjoint sets of measure 2^{-n} such that $I(k, n-1) = I(2k, n) \cup I(2k+1, n)$. Choose a sequence of positive integers $\{p_n\}$ with the following properties:

$$p_1 \geqq 3, \tag{1}$$

$$|a_{p_n}| < 2^{-n}, \tag{2}$$

$$p_{2^n - n} > q_n, \tag{3}$$

where $\{q_n\}$ denotes the sequence of positive integers complementary to $\{p_n\}$. We can assume that $\{q_n\}$ is infinite since otherwise $\sum a_n^2 < \infty$ and $\sum a_n \phi_n$ converges a.e. for every H-system.

Let $\phi_1 = 1$ and assume $\{\phi_k\}$ has been defined for $k \leqq m-1$ in such a way that $\sigma(\phi_1, \cdots, \phi_{m-1})$ is composed of atoms $I(k_1, n_1)$, $I(k_2, n_2)$, $\cdots$, $I(k_{m-2}, n_{m-2})$, $I(2^q - 1, q)$, where $n_1, n_2, \cdots, n_{m-2}, q$ are dependent on m with $n_1 \leqq n_2 \leqq \cdots \leqq n_{m-2}$ and $I(2^q - 1, q)$ is the unique atom that can be so expressed. If $m = p_n$ for some n, define

$$\begin{aligned} \phi_m &= 2^{n_1/2} && \text{on } I(2k_1, n_1+1), \\ &= -2^{n_1/2} && \text{on } I(2k_1+1, n_1+1), \\ &= 0 && \text{otherwise.} \end{aligned}$$

If $m \neq p_n$ for any n, then define ϕ_m similarly on $I(2^q-1, q)$. It is easily seen that $\{\phi_m\}$ is an H-system.

The sequence of atoms of the form $I(2^q-1, q)$ used in the construction form a decreasing sequence of sets and, since $\{q_n\}$ is infinite, their intersection, call it Q, has measure zero. Given x, x not in Q, choose M so large that the atom of $\sigma(\phi_1, \cdots, \phi_M)$ of the form $I(2^q-1, q)$ does not contain x. It can be seen from the definition of $\{\phi_m\}$, for $2^j - j \leqq n < 2^{j+1}-(j+1)$ and $m = p_n$, that μ (support $\phi_m) = 2^{-j}$. Therefore, using the notation of Theorem 2,

$$\sum_{n_k > M} a_{n_k}^2 \epsilon_k \leqq \sum_{p_n > M} a_{p_n}^2 2^j \leqq \sum \left(\frac{1}{2^j}\right)^2 2^j < \infty.$$

By Theorem 2, $\sum a_k \phi_k$ converges a.e.

The proof of part (b) follows by an identical construction except that $\{a_{p_n}\}$ is chosen as a subsequence of $\{a_n\}$ which is bounded away from zero.

It is known [7] that there exist rearrangements of the Haar system such that the corresponding expansion of some function in $L^2(0, 1)$ diverges a.e. As corollaries to the above constructions in Theorem 3, we can compare this to rearrangements of the Haar system which preserve the convergence a.e. of expansions of integrable functions.

COROLLARY. *If $\{a_k\}$ is a sequence such that* $\liminf |a_k| = 0$, *then there exists a rearrangement of the Haar system, $\{\phi_n\}$, which is an H-system, for which $\sum a_k \phi_k$ converges a.e.*

COROLLARY. *If $\{a_k\}$ is a sequence such that* $\limsup |a_k| > 0$, *then there exists a rearrangement of the Haar system, $\{\phi_n\}$, which is an H-system, for which $\sum a_k \phi_k$ diverges a.e.*

PROOF. If the sets $I(k, n)$ are chosen to be the intervals $(k/2^n, (k+1)/2^n)$, then the orthonormal system constructed in Theorem 3 is a rearrangement of the Haar system and the corollaries then follow.

5. Sets of completeness for H-systems.

In this section we will assume that μ is Lebesgue measure on $[0, 1]$.

An H-system need not be complete on $L^2(0, 1)$. For example, the

σ-field generated by $\{\phi_n\}$, denoted by $\sigma(\phi_1, \phi_2, \cdots)$, could contain atoms. In [2], Price and Zink characterized the subsets of $[0, 1]$ on which a subsystem of the classical Haar system is complete.

Let G be a measurable subset of $[0, 1]$. A system of functions $\{\mathfrak{F}\}$ is *total in measure* on G if for every measurable function f on G there is a sequence of finite linear combinations of functions of $\{\mathfrak{F}\}$ that converges in measure to f.

The following theorems of Talalyan compare totality in measure with completeness.

THEOREM A. *A system of functions $\{\mathfrak{F}\}$ is total in measure on a measurable subset G of $[0, 1]$ if and only if to each $\epsilon>0$, there corresponds a measurable subset E of G such that $\mu(E) \geqq \mu(G)-\epsilon$ and $\{\mathfrak{F}\}$ is complete in $L^2(E)$.*

THEOREM B. *If a sequence of measurable functions is total in measure on a measurable subset of $[0, 1]$, then it remains so when a finite set of functions is removed from it.*

For the proofs, see [8] and [9].

Given a sequence of measurable functions $\{f_n\}$, let the lim sup (support f_n) be called the *support* of $\{f_n\}$. Two measurable sets will be called equivalent if they differ by sets of measure zero.

THEOREM 4. *A subsystem $\{\phi_n\}$ of an H-system is total in measure on a measurable set G in $[0, 1]$ if and only if G is equivalent to a set in $\sigma(\phi_1, \phi_2, \cdots)$ which is contained in the support of $\{\phi_n\}$.*

PROOF. If $\{\phi_n\}$ is total in measure on G then χ_G is the limit in measure of a sequence of functions, $\{f_n\}$, where each f_n is measurable with respect to $\sigma(\phi_1, \cdots, \phi_n)$ and hence G is equivalent to a set in $\sigma(\phi_1, \phi_2, \cdots)$. By Theorem B, χ_G equals a.e. a function measurable with respect to $\sigma(\phi_{n+1}, \phi_{n+2}, \cdots)$. This implies almost all x in G are in the support of at least one ϕ_k, $k \geqq n+1$ and, hence, in the lim sup (support ϕ_n).

Conversely, if G is in $\sigma(\phi_1, \phi_2, \cdots)$ and in the support of $\{\phi_n\}$, then $f_n = E(\chi_G | \phi_1, \cdots, \phi_n)$ converges to χ_G a.e. by the martingale convergence theorem [5, p. 331]. Since each $\sigma(\phi_1, \cdots, \phi_n)$ is composed of a finite collection of atoms, each f_n is a finite linear combination of $\phi_1, \cdots, \phi_n$ and so $\{\phi_n\}$ is total in measure on G.

REMARK. Theorem 4 will hold for every measurable subset of the support of $\{\phi_n\}$ if $\sigma(\phi_1, \phi_2, \cdots)$ restricted to the support of $\{\phi_n\}$ is equivalent to the Borel σ-algebra. That this happens if $\{\phi_n\}$ is a subsystem of the Haar functions is a result of Price and Zink [2] and we obtain it as a corollary.

COROLLARY. *Let* $\{\phi_n\}$ *be a subsystem of the Haar functions.* $\{\phi_n\}$ *is total in measure on a set G if and only if almost all points of G are contained in the support of* $\{\phi_n\}$.

PROOF. Let E denote the support of $\{\phi_n\}$ and let

$$\psi_n = E(\chi_G \mid \phi_1, \cdots, \phi_n).$$

Since the support of each ϕ_n is an interval, the atoms of $\sigma(\phi_1, \cdots, \phi_n)$ are intervals. Given a point x, let $I_n(x)$ denote the interval of $\sigma(\phi_1, \cdots, \phi_n)$ containing x. For points in E, $\lim_{n\to\infty} \mu(I_n(x))=0$. From the definition of conditional expectation, $\psi_n(x)=\mu(G\cap I_n(x))/\mu(I_n(x))$, and the martingale convergence theorem implies

$$\lim_{n\to\infty} \psi_n = E(\chi_G \mid \phi_1, \phi_2, \cdots) \text{ a.e.}$$

To see that $\lim_{n\to\infty}\psi_n=\chi_G$ a.e., we observe that on the set where $\lim_{n\to\infty}\mu(I_n(x))=0$, $\psi_n(x)$ converges, except for a set of measure zero, to 1 for x in G and to 0 for x not in G by the density theorem; while on the set where $\lim_{n\to\infty}\mu(I_n(x))=\alpha$, for some $\alpha>0$, $\lim_{n\to\infty}\mu(I_n(x)\cap G)=0$ since $\lim_{n\to\infty} I_n(x)\cap G=\emptyset$. Therefore, G is equivalent to a set in $\sigma(\phi_1, \phi_2, \cdots)$.

REFERENCES

1. R. F. Gundy, *Martingale theory and pointwise convergence of certain orthogonal series*, Trans. Amer. Math. Soc. **124** (1966), 228–248.
2. J. J. Price and R. E. Zink, *On sets of completeness for families of Haar functions*, Trans. Amer. Math. Soc. **119** (1965), 262–269.
3. J. J. Price, *Orthonormal sets with non-negative Dirichlet kernels*, Trans. Amer. Math. Soc. **95** (1960), 256–262.
4. L. Leindler, *Über Konvergenz- und Summationeigenschaften von Haaren Reihen*, Acta. Math. Sci. (Szeged) **26** (1965), 19–30.
5. J. L. Doob, *Stochastic processes*, Wiley, New York, 1953.
6. G. Alexits, *Convergence problems of orthogonal series*, Pergamon Press, New York, 1961.
7. P. L. Ulyanov, *Divergent fourier series*, Uspehi Mat. Nauk. **16** (1961), 61–142.
8. A. A. Talalyan, *On the convergence a.e. of subsequences of partial sums of general orthogonal series*, Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **10** (1957), 17–34.
9. C. Goffman and D. Waterman, *Basic sequences in the space of measurable functions*, Proc. Amer. Math. Soc. **11** (1960), 211–213.

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