ON GRONWALL'S INEQUALITY

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Let the functions u and h be continuous and nonnegative on the interval $[0, 1]$; and let $c \geqq 0$ be a constant. The classical Gronwall inequality [1] states: if

$$u(x) \leqq c + \int_0^x h(y)u(y)dy, \qquad 0 \leqq x \leqq 1, \tag{1}$$

then

$$u(x) \leqq c \exp\left(\int_0^x h(y)dy\right), \qquad 0 \leqq x \leqq 1. \tag{2}$$

Various linear generalizations of this inequality have been given; see, for example, [2, p. 37], [3], and [4]. In most of these cases, the upper bound for u is just the solution of the *equation* corresponding to the integral inequality of the type (1). That is, such results are essentially comparison theorems. An abstract version of this type of comparison theorem, using lattice-theoretic methods, has been given in [5].

Presented below is a generalization of the Gronwall inequality, which contains the previous results concerning integral inequalities of the type (1).

THEOREM. *Let the functions u and f be continuous on the interval $[0, 1]$; let the function K be continuous and nonnegative on the triangle $0 \leqq y \leqq x \leqq 1$. If*

$$u(x) \leqq f(x) + \int_0^x K(x, y)u(y)dy, \qquad 0 \leqq x \leqq 1, \tag{3}$$

then

$$u(x) \leqq f(x) + \int_0^x H(x, y)f(y)dy, \qquad 0 \leqq x \leqq 1,$$

where $H(x, y) = \sum_{i=1}^{\infty} K_i(x, y)$, $0 \leqq y \leqq x \leqq 1$, is the resolvent kernel, and the K_i $(i = 1, 2, \cdots)$ are the iterated kernels of K.

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PROOF. From (3), one has

$$u(x) \leqq f(x) + \int_0^x K(x, y)f(y)dy + \int_0^x K(x, y) \int_0^y K(y, z)u(z)dzdy$$

$$= f(x) + \int_0^x K_1(x, y)f(y)dy + \int_0^x K_2(x, y)u(y)dy,$$

for $0 \leqq x \leqq 1$. The remainder of the proof is by induction and a standard estimation procedure showing the resulting series to be uniformly convergent.

The previous results, in which an explicit upper bound for u was obtained, are merely those cases for which the resolvent kernel H can be summed in "closed form." For example, if $K(x, y) = g(x)h(y) \geqq 0$, $0 \leqq y \leqq x \leqq 1$, then

$$H(x, y) = \sum_{i=1}^{\infty} \frac{g(x)h(y)}{(i-1)!} \left(\int_y^x g(z)h(z)dz \right)^{i-1}$$

$$= g(x)h(y) \exp\left(\int_y^x g(z)h(z)dz \right),$$

since one can show by induction that each $K_i (i = 1, 2, \cdots)$ is given by the appropriate term in the sum for H.

REFERENCES

1. T. H. Gronwall, *Note on the derivatives with respect to a parameter of the solutions of a system of differential equations*, Ann. of Math. (2) **20** (1919), 292–296.

2. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.

3. G. S. Jones, *Fundamental inequalities for discrete and discontinuous functional equations*, J. Soc. Indust. Appl. Math. **12** (1964), 43–57.

4. D. Willett, *A linear generalization of Gronwall's inequality*, Proc. Amer. Math. Soc. **16** (1965), 774–778.

5. D. L. Hanson and Paul Waltman, *A note on a functional equation*, J. Math. Anal. Appl. **10** (1965), 330–333.

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