BOUNDED POLYMEASURES AND ASSOCIATED TRANSLATION COMMUTATIVE POLYNOMIAL OPERATORS

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1. **Introduction.** Morse and Transue [6–8] defined and developed properties of bimeasures, quadratic generalizations of the linear Radon measures. This paper is devoted to a polynomial generalization in the bounded case, the polymeasure, and to indicating that bounded polymeasures provide a means of representing translation commutative polynomial operators on the space $C(G)$ of bounded continuous functions on a locally compact group G when the operators are strictly bounded and strictly continuous on strictly bounded sets. In the linear case the result demonstrated is; a left translation invariant operator on $C(G)$ takes the form $y(h)=\int x(hg)d\Lambda(g)$ with Λ a bounded Radon measure, $x(g), y(g)\in C(G)$, if and only if the operator is strictly bounded and strictly continuous on strictly bounded sets. The desire to obtain a representation for translation invariant polynomial operators on function space results from a growing use of such operators in stochastic process theory, see [5], [9].

2. **Bounded polymeasures.** Throughout the paper bold face letters $\boldsymbol{f}$, $\boldsymbol{m}$, $\boldsymbol{p}$ will denote k-vectors. If X is locally compact $C(X)$ will denote the set of bounded, complex-valued, continuous functions on X, $C_0(X)$ will denote the f in $C(X)$ which vanish at infinity, $C_{00}(X)$ will denote the f in $C(X)$ with compact support. If there is no confusion as to the basic space X we will write C, C_0, C_{00}. For real-valued functions f, g on a set X, the expression $f\leqq g$ will mean $f(x)\leqq g(x)$ for all x in X, while for k-vectors $\boldsymbol{f}\leqq\boldsymbol{g}$ will mean $f_i\leqq g_i$ for each coordinate. If A is a linear space, A^k will denote the direct product of A with itself k times. When A has a topology, we will consider two sorts of continuity in A^k, continuity separately in each coordinate and joint continuity in all coordinates. The norm $\|f\|$ on C, C_0, and C_{00} will denote the supremum norm.

Buck [1] introduced the strict topology on $C(X)$. It is a locally convex topology defined by the family of seminorms $\|m\|_\phi=\sup|m(x)\phi(x)|$, for m in C and ϕ in C_0. The topology is complete, its bounded sets are the bounded sets of the uniform topology and on a bounded set the topology coincides with the uniform on compacta topology [2].

Received by the editors March 4, 1966.

DEFINITION. A bounded kth-order polymeasure is a uniformly continuous, multilinear, complex-valued functional on $C_{00}(X)^k$ where X is a locally compact space. Its values will be denoted by $\Lambda(\boldsymbol{f})$ where $\boldsymbol{f}$ is in C_{00}^k. The uniform bound will be denoted $\|\Lambda\|$.

We note that if $k=1$, Λ is the usual Radon integral, while if $k=2$, Λ is the C-bimeasure considered in [6–8].

Suppose $\boldsymbol{p}\geqq 0$ is in $C(X)^k$. We define, for Λ a bounded polymeasure,

$$\Lambda^*(\boldsymbol{p}) = \sup_{|\boldsymbol{f}|\leqq \boldsymbol{p}} |\Lambda(\boldsymbol{f})|$$

for $\boldsymbol{f}$ in C_{00}^k. By elementary calculations we see that,

(a) $\Lambda^*(\boldsymbol{p})\leqq\Lambda^*(\boldsymbol{q})$ if $\boldsymbol{p}\leqq\boldsymbol{q}$,

(b) $\Lambda^*(a_1p_1, \cdots, a_kp_k)=a_1\cdots a_k\Lambda^*(p_1, \cdots, p_k)$ for real $a_j\geqq 0$,

(c) $\Lambda^*(p_1+q_1, p_2, \cdots, p_k)\leqq\Lambda^*(p_1, \cdots, p_k)+\Lambda^*(q_1, p_2, \cdots, p_k)$ with a similar result for the second, third, ... arguments of Λ^*, $q_1\geqq 0$,

(d) $\Lambda^*(\boldsymbol{p})\leqq\|\Lambda\|\,\|p_1\|\cdots\|p_k\|$,

(e) $\Lambda^*(\boldsymbol{1})=\|\Lambda\|$.

Let us now proceed to an extension of Λ to C^k.

THEOREM 1. *A bounded kth-order polymeasure Λ on C_{00}^k has a unique extension $\bar{\Lambda}$ to C^k such that*

(i) *$\bar{\Lambda}$ is strictly continuous in each of its arguments,*

(ii) $|\bar{\Lambda}(m_1, \cdots, m_k)|\leqq\Lambda^*(|m_1|, \cdots, |m_k|)$,

(iii) *$\bar{\Lambda}$ is jointly uniformly continuous and*

(iv) *$\bar{\Lambda}$ is jointly strictly continuous on strictly bounded sets.*

PROOF. We extend Λ by recursion. For $j=0, \cdots, k-1$ in turn, let Λ_j be uniformly continuous and multilinear on $L_j=C^j\times C_{00}^{k-j}$, then $m_{j+1}\to\Lambda_j(m_1, \cdots, m_{j+1}, \cdots, m_k)$ is a uniformly continuous linear functional on C_{00} with norm $\|\Lambda_j\|\,\|m_1\|\cdots\|m_j\|\,\|m_{j+2}\|\cdots\|m_k\|$ and is thus a Radon measure. It therefore (Buck [2]) has a unique extension to C which is continuous in the strict topology and retains its bound. Thus we get a multilinear functional defined on L_{j+1} with normal equal to $\|\Lambda_j\|$. Λ_k is the required extension, $\bar{\Lambda}$, of $\Lambda=\Lambda_0$ to C^k. To see this we first note that $\bar{\Lambda}$ is strictly continuous in each of its arguments and that it does not depend on the order in which the C_{00}'s are changed into C's for we may take any pair of coordinates and allow only these two to vary and we have a bimeasure and its usual extension (Theorem 11.1 of [8]).

(ii) follows from the definition of Λ^* and the fact that we may evaluate $\bar{\Lambda}(m_1, \cdots, m_k)$ as

$$\lim_{\alpha_1}\cdots\lim_{\alpha_k}\Lambda(m_{1\alpha_1}, \cdots, m_{k\alpha_k})$$

where $m_{j\alpha_j}$ is a generalized sequence of elements of C_{00} that converges to m_j in the strict topology.

(iii) follows from (ii) and since $\Lambda^*(|m_1|, \cdots, |m_k|) \leqq \|\Lambda\| \, \|m_1\| \cdots \|m_k\|$.

To prove (iv) we first note that an obvious extension of Theorem 6.1 of [7] indicates the existence of e_j in C_{00}, $0 \leqq e_j \leqq 1$ such that $\Lambda^*(1, \cdots, 1, 1-e_j, 1, \cdots, 1)$ is as small as desired. Now consider $\{\boldsymbol{m}_\alpha\}_{\alpha \in \Delta}$ a uniformly bounded generalized sequence of elements of C^k, $\|m_{j\alpha}\| \leqq M_j$, converging to $\boldsymbol{m}$ in the joint topology of uniform convergence on compacta. e_j has compact support; therefore there exists α_0 such that for $\alpha > \alpha_0$, $\|(m_j - m_{j\alpha})e_j\|$ is as small as desired.

$$(2.1)\qquad \begin{aligned} &|\bar{\Lambda}(\boldsymbol{m}) - \bar{\Lambda}(\boldsymbol{m}_\alpha)| \\ &\quad \leqq \left|\sum \bar{\Lambda}(m_1, \cdots, m_{j-1}, m_j - m_{j\alpha}, m_{j+1\alpha}, \cdots, m_{k\alpha})\right| \\ &\quad \leqq \sum \Lambda^*(M_1, \cdots, M_{j-1}, |m_j - m_{j\alpha}|, M_{j+1}, \cdots, , M_k). \end{aligned}$$

Now $\Lambda^*(M_1, \cdots, M_{j-1}, |m_j - m_{j\alpha}|, M_{j+1}, \cdots, M_k)$ is less than or equal a multiple of

$$\|(m_j - m_{j\alpha})e_j\| + \|m_j - m_{j\alpha}\| \Lambda^*(1, \cdots, 1, 1 - e_j, 1, \cdots, 1).$$

Using the uniform boundedness of the sequence $\{\boldsymbol{m}_\alpha\}_{\alpha \in \Delta}$ we see that (2.1) may be made as small as desired and $\bar{\Lambda}(\boldsymbol{m}_{j\alpha}) \to \bar{\Lambda}(\boldsymbol{m})$, proving (iv).

The extension $\bar{\Lambda}$ is seen to be unique and will also be called a bounded kth-order polymeasure.

3. Translation commutative polynomial operators.

In this section a 1-1 correspondence will be set up between bounded polymeasures on a locally compact group G and kth degree, left translation commutative, monomial operators on $C(G)$ which are strictly bounded and strictly continuous on strictly bounded sets.

Polynomial operators are discussed in Hille and Philips [3] and Mazur and Orlicz [4] for example. Given linear spaces $\mathfrak{X}$ and $\mathfrak{Y}$ a map $P: \mathfrak{X} \to \mathfrak{Y}$ is defined to be a polynomial of degree k if

$$\Delta_u^{k+1} P(x) = 0$$

for all x, u in $\mathfrak{X}$ where $\Delta_u P(x) = P(x+u) - P(x)$ and 0 is the zero of $\mathfrak{Y}$.

A polynomial map of degree k is a monomial map of degree k if $M(\alpha x) = \alpha^k M(x)$ for all scalars α. One may associate uniquely with each monomial map of degree k a kth-order multilinear map $N: \mathfrak{X}^k \to \mathfrak{Y}$ called the polar. A polynomial map of degree k may be written as the sum of monomials of degrees less than or equal to k. These details may be found in [3].

It is important to note that when $\mathfrak{X}$ and $\mathfrak{Y}$ are linear topological spaces, a monomial map M is continuous if and only if its polar is continuous. (This result is Theorem 26.2.6 of [3] for Banach spaces. The needed result is a straightforward generalization.)

Henceforth considerations will be limited to left translations of elements of the group G. Given functions $m(g)$ on G the translation operator, U_h, is defined by $U_h m(g) = m(hg)$, h in G. e is the identity of G.

THEOREM 2. *The pair of relations* (i) $\Omega(m) = M(m)(e)$, (ii) $M(m)(g)$ $\Omega(U_g m)$ *set up a* 1-1 *correspondence between kth-order, left translation commutative, monomial operators* $M: C(G) \to C(G)$ *and kth-order monomial functions* $\Omega: C(G) \to$ *complex numbers, such that if* Ω *is strictly continuous on strictly bounded sets, then* M *is strictly bounded and strictly continuous on strictly bounded sets and conversely.*

PROOF. The proof of a 1-1 correspondence and the stated continuity is identical with the corresponding proof in [1].

We conclude the proof by noting that (i) and (ii) do in fact define monomials under the stated assumptions.

THEOREM 3. *A kth-order monomial operator* $M: C(G) \to C(G)$ *is strictly bounded and strictly continuous on strictly bounded sets if and only if it has the form*

$$M(m)(g) = \Lambda(U_g m, \cdots, U_g m) \tag{3.1}$$

where Λ *is a bounded kth-order polymeasure and* m *belongs to* $C(G)$. *An* M *given by* (3.1) *is also continuous in the uniform topology and has as bound* $\|\Lambda\|$, *the bound of the polymeasure* Λ.

PROOF. The theorem follows directly. Theorem 2 indicates the existence of a monomial Ω with certain continuity properties. Take Λ to be the polar of Ω. The continuity properties that Λ inherits make it a bounded polymeasure. (3.1) now follows from the relation between a monomial and its polar (see [3] for this last).

It was noted earlier that a polynomial of order k may be expressed as the sum of monomials of orders less than or equal to k. These monomials are continuous if the polynomial is and conversely. (This results from the fact that the monomials are linear combinations of values of the polynomial.) We now see

COROLLARY. *A kth order polynomial operator* $P: C(G) \to C(G)$ *is* *strictly bounded and strictly continuous on strictly bounded sets if and only if it has, for m in C(G), the form*

$$P(m)(g) = \sum_{j=0}^{k} \Lambda_j(U_g m, \cdots, U_g m)$$

where Λ_j is a bounded jth-order polymeasure.

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