ON THE CLOSURE OF THE NUMERICAL RANGE OF AN OPERATOR

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If T is a bounded linear mapping (briefly, operator) in a Hilbert space $\mathcal{H}$, the *numerical range* of T is the set $W(T) = \{(Tx, x): \|x\| = 1\}$; thus $W(T)$ is convex [8, p. 131], and its closure $\mathrm{cl}[W(T)]$ is compact and convex. Roughly speaking, in this note we observe that $\mathrm{cl}[W(T)]$ can be uniquely defined for an element T of an abstract C^*-algebra, while $W(T)$ cannot. The C^*-algebra setting yields an extension of the spectral convexity theorem [8, p. 327] to nonnormal operators (Corollary 1 of Theorem 3), as well as a reformulation of a theorem of C. Berger (Corollary 2 of Theorem 3).

1. **An algebraic characterization of the closure of the numerical range.** Let T be an operator in a Hilbert space $\mathcal{H}$. Our objective in this section is to describe $\mathrm{cl}[W(T)]$ in algebraic terms; since $\mathrm{cl}[W(T)]$ is the intersection of all closed half planes H containing it, it is sufficient to describe those H that contain $W(T)$ (see Theorem 1). We write $\mathrm{Re}\,\lambda$ for the real part of a complex number λ; analogously we define $\mathrm{Re}\,T = \frac{1}{2}(T^* + T)$.

LEMMA 1. *$\mathrm{Re}\,T \geqq 0$ if and only if $(T - \alpha I)^*(T - \alpha I) \geqq \alpha^2 I$ for all $\alpha < 0$.*

PROOF. For any real number α,

$$(T - \alpha I)^*(T - \alpha I) - \alpha^2 I = T^*T - \alpha(T^* + T).$$

If the left side of this equation is $\geqq 0$ for all $\alpha < 0$, then for all $\alpha < 0$ we have $\alpha(T^* + T) \leqq T^*T$, $T^* + T \geqq (T^*T)/\alpha$, and $T^* + T \geqq 0$ results on letting $\alpha \to -\infty$; thus $\mathrm{Re}\,T \geqq 0$. If, conversely, $\mathrm{Re}\,T \geqq 0$, then for any $\alpha < 0$ we have $T^* + T \geqq 0 \geqq (T^*T)/\alpha$, and the steps of the above argument may be reversed.

LEMMA 2. *Let $H_0 = \{\lambda: \mathrm{Re}\,\lambda \geqq 0\}$. Then $W(T) \subset H_0$ if and only if $(T - \lambda I)^*(T - \lambda I) \geqq (\mathrm{Re}\,\lambda)^2 I$ for all λ not in H_0.*

PROOF. Since $((\mathrm{Re}\,T)x, x) = \mathrm{Re}(Tx, x)$, clearly $W(T) \subset H_0$ if and only if $\mathrm{Re}\,T \geqq 0$. If $\lambda = \alpha + i\beta$ with α and β real, then $T - \lambda I = (T - i\beta I) - \alpha I$, and $\mathrm{Re}\,T = \mathrm{Re}(T - i\beta I)$. Fixing any real β, by Lemma 1 we have $(T - \lambda I)^*(T - \lambda I) \geqq \alpha^2 I$ for all $\alpha < 0$ if and only if $\mathrm{Re}(T - i\beta I) \geqq 0$, that is, $\mathrm{Re}\,T \geqq 0$.

If H is a set of complex numbers, we write

Received by the editors June 7, 1966.

$$\text{dist}(\lambda, H) = \inf\{ |\lambda - \mu| : \mu \in H\}.$$

THEOREM 1. *Let H be a closed half plane of the complex plane. Then* $W(T) \subset H$ *if and only if* $(T-\lambda I)^*(T-\lambda I) \geqq [\text{dist}(\lambda, H)]^2 I$ *for all complex numbers* λ.

PROOF. Obviously we need consider only λ not in H. The case $H = H_0$ is covered by Lemma 2. In general, we have $f(H) = H_0$ for a suitable linear function $f(\lambda) \equiv \mu\lambda + \tau$ with $|\mu| = 1$. Let $S = f(T) = \mu T + \tau I$. Evidently $W(S) = f[W(T)]$, thus $W(T) \subset H$ if and only if $W(S) \subset H_0$. Also $S - f(\lambda)I = \mu(T - \lambda I)$, hence $(S-f(\lambda)I)^*(S-f(\lambda)I) = (T-\lambda I)^*(T-\lambda I)$. Finally, $\text{dist}(f(\lambda), H_0) = \text{dist}(f(\lambda), f(H)) = \text{dist}(\lambda, H)$. Thus the general case follows on applying the special case $H = H_0$ to the operator S.

Writing $\sigma(T)$ for the spectrum of T, we may reformulate Theorem 1 in terms of resolvents:

COROLLARY. *Let H be a closed half plane of the complex plane. Then* $W(T) \subset H$ *if and only if*

(i) $\sigma(T) \subset H$, *and*

(ii) $\|(T-\lambda I)^{-1}\| \leqq 1/\text{dist}(\lambda, H)$ *for all* λ *not in* H.

PROOF. If $W(T) \subset H$, then $\sigma(T) \subset \text{cl}[W(T)] \subset H$ (cf. [2, Lemma 1]), and so (i) holds. On the other hand if λ is not in $\sigma(T)$ or in H, the inequality in (ii) is equivalent to

$$\text{dist}(\lambda, H)\|(T - \lambda I)^{-1}y\| \leqq \|y\|$$

for all vectors y, that is,

$$\text{dist}(\lambda, H)\|x\| \leqq \|(T - \lambda I)x\|$$

for all vectors x, that is,

$$(T - \lambda I)^*(T - \lambda I) \geqq [\text{dist}(\lambda, H)]^2 I.$$

The corollary now follows at once from Theorem 1.

2. C^***-algebras.** Let A be a C^*-algebra with unity 1 [4, p. 6]. By the Gel'fand-Naĭmark theorem there exists a faithful $*$-representation $a \rightarrow T_a$ of A as operators on a suitable Hilbert space, with $T_1 = I$ [4, p. 39; 6, p. 244]. If $a \in A$, we define the *closed numerical range* of a, denoted $\overline{W}(a)$, to be the set $\text{cl}[W(T_a)]$; the definition is justified by the following theorem:

THEOREM 2. *Let A be a* C^**-algebra with unity, and suppose we are given faithful* $*$*-representations* $a \rightarrow T_a$ *and* $a \rightarrow S_a$, *as operators on Hilbert spaces* $\mathcal{H}$ *and* $\mathcal{K}$, *respectively, such that* $T_1 = I$ *and* $S_1 = I$. *Then* $\text{cl}[W(T_a)] = \text{cl}[W(S_a)]$ *for all a in A.*

PROOF. The systems $\{T_a: a\in A\}$ and $\{S_a: a\in A\}$ are C^*-algebras of operators [4, p. 16], and $T_a\to S_a$ is a $*$-isomorphism of one onto the other; in particular, $T_a\to S_a$ preserves positivity [4, pp. 8 and 12]; [7, §§104 and 118]. Fix a in A. If H is a closed half plane, it is clear from Theorem 1 that $\mathrm{cl}[W(T_a)]\subset H$ if and only if $\mathrm{cl}[W(S_a)]\subset H$. Since a closed convex set is the intersection of the closed half planes that contain it, the theorem is proved.

Let us write Σ for the set of all normalized states of A, that is, the set of all linear forms f on A such that $f(1)=1$ and $f(a^*a)\geqq 0$ for all a in A; then Σ is a convex subset of the dual space of A, and is compact in the weak* topology (cf. [4, p. 37]; [6, p. 222, Lemma 4.6.2]). For any a in A we write $\Sigma(a)=\{f(a): f\in\Sigma\}$. Since the mapping $f\to f(a)$ is linear and weak* continuous, $\Sigma(a)$ is compact and convex; indeed, we have the following intrinsic characterization of $\overline{W}(a)$:

THEOREM 3. *If A is a C^*-algebra with unity, and Σ is the set of all normalized states of A, then*

$$\Sigma(a) = \overline{W}(a) \tag{1}$$

for every a in A.

PROOF. Each f in Σ leads to a canonical $*$-representation of A on a Hilbert space $\mathcal{H}_f$, and by the Gel′fand-Naĭmark theorem the direct sum of these representations is a faithful $*$-representation $a\to T_a$ of A as operators on the direct sum $\mathcal{H}$ of the Hilbert spaces $\mathcal{H}_f$ [4, p. 39]; [6, p. 197]. Fix a in A. It is obvious from the construction that $f(a) \in W(T_a)$ for each f in Σ, thus $\Sigma(a)\subset W(T_a)$. On the other hand if $x\in\mathcal{H}$, $\|x\|=1$, then the formula $f(b)=(T_bx, x)$ defines a normalized state of A, and $(T_ax, x)=f(a)\in\Sigma(a)$; varying x we have $W(T_a)\subset\Sigma(a)$, and therefore $\Sigma(a)=W(T_a)$. In particular, for each a in A, $W(T_a)$ is closed and therefore coincides with $\overline{W}(a)$. (This precludes an invariant definition of "numerical range" in the C^*-algebra setting; the numerical range can be closed in one representation and not closed in another.)

Denote the set of extremal points of Σ by Σ_e; these are precisely the normalized pure states of A, that is, the elements of Σ for which the canonical $*$-representation of A is irreducible [4, p. 37]; [6, p. 223]. For any a in A we write $\Sigma_e(a)=\{f(a): f\in\Sigma_e\}$. By the Kreĭn-Mil′man theorem, Σ is the weak* closure of the convex hull of Σ_e; it follows readily that for each a in A, $\Sigma(a)$ is the closure of the convex hull of $\Sigma_e(a)$. Citing formula (1) we have

COROLLARY 1. *If A is a C^*-algebra with unity, and Σ_e is the set of all normalized pure states of A, then*

$$\text{cl}[\text{conv}\,\Sigma_e(a)] = \overline{W}(a) \tag{2}$$

for every a in A.

If, in particular, A is commutative (and so every a in A is normal), then Σ_e coincides with the set of characters of A [**4**, p. 23, B3]; [**6**, p. 229] and one has $\Sigma_e(a)=\sigma(a)$ for all a in A, where $\sigma(a)$ denotes the spectrum of a. Since the convex hull of $\sigma(a)$ is closed (cf. [**2**]) formula (2) yields

$$\text{conv}\,\sigma(a) = \overline{W}(a)$$

for all a in A; thus formula (2) may be viewed as a noncommutative extension of the classical spectral convexity theorem for normal operators [**8**, p. 327]; [**2**].

COROLLARY 2. *With notations as in Theorem 3, define $\omega(a) = \sup\{|f(a)| : f\in\Sigma\}$. Then $\omega(a^n)\leqq(\omega(a))^n$ for all positive integers n.*

PROOF. If T is any operator, write $\omega(T)=\sup\{|\lambda| : \lambda\in W(T)\}$. By a theorem of C. Berger [**3**], $\omega(T^n)\leqq(\omega(T))^n$ for all n, and the corollary follows at once from Theorem 3. In particular, if a is in the polar set of Σ (that is, if $\omega(a)\leqq 1$), then so is a^n; Berger's result is thus seen to be equivalent to the assertion that the polar set of Σ is closed under the formation of powers.

COROLLARY 3. *If $\mathcal{H}$ is a Hilbert space and f is any normalized state on the C^*-algebra $\mathcal{L}(\mathcal{H})$ of all operators in $\mathcal{H}$, then for each T in $\mathcal{L}(\mathcal{H})$ there exists a sequence (depending on T) of unit vectors x_n in $\mathcal{H}$ such that $f(T)=\lim(Tx_n, x_n)$.*

PROOF. Applying Theorem 3 to the C^*-algebra $A=\mathcal{L}(\mathcal{H})$, we have $f(T)\in\overline{W}(T)$ by formula (1); but $\overline{W}(T)=\text{cl}[W(T)]$ since the identity representation may be employed in the definition of $\overline{W}(T)$. We remark that the proof of Theorem 3 yields a canonical faithful $*$-representation $T\rightarrow T^\#$ of $\mathcal{L}(\mathcal{H})$ on a suitable Hilbert space $\mathcal{K}$, in such a way that $W(T^\#)$ is closed for every T in $\mathcal{L}(\mathcal{H})$, indeed, $W(T^\#)=\text{cl}[W(T)]$. We now present a highly noncanonical representation of this sort.

3. **Approximate proper vectors.** With the notations of [**1**], let $\mathcal{H}$ be a Hilbert space, $\mathcal{K}$ the extension of $\mathcal{H}$ constructed from a generalized limit, and $T\rightarrow T^\circ$ the resulting faithful $*$-representation of $\mathcal{L}(\mathcal{H})$ on $\mathcal{K}$.

PROPOSITION. *For any operator T in $\mathcal{H}$, $W(T^\circ)$ is closed; indeed, $W(T^\circ)=\text{cl}[W(T)]$.*

PROOF. We recall from [1] the elementary observations that $I^\circ = I$, and $T^\circ \geqq 0$ if and only if $T \geqq 0$. It follows that $\text{cl}[W(T^\circ)] = \text{cl}[W(T)]$ (see the proof of Theorem 2). We conclude the proof by showing that $\text{cl}[W(T)] \subset W(T^\circ)$. Suppose $\lambda = \lim(Tx_n, x_n)$, where x_n is a sequence of unit vectors in $\mathcal{H}$. Writing $u = \{x_n\}'$ as in [1], we have $\|u\| = 1$ and $(T^\circ u, u) = \text{glim}(Tx_n, x_n) = \lim(Tx_n, x_n) = \lambda$.

ADDED IN PROOF. E. Berkson has pointed out to us that Theorem 3 is implicit in an article of G. Lumer [Trans. Amer. Math. Soc. **100** (1961), 29–43, Theorem 11] since, on a C^*-algebra with unity, a continuous linear form f is positive if and only if $f(1) = \|f\|$ [H. F. Bohnenblust and S. Karlin, Ann. of Math. **62** (1955), 217–229, Theorem 10] (see also p. 25 of [1]). Theorem 3 is also covered by Theorem 12 of Bohnenblust and Karlin [loc. cit.]. An elementary proof of Berger's result has been published by C. Pearcy [Michigan Math. J. **13** (1966), 289–291].

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