ON THE PROXIMAL RELATION BEING CLOSED

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1. Introduction. In this paper, we utilize the algebraic techniques suggested in [2] and [3] to investigate some questions about the proximal relation being closed, where the transformation groups have compact Hausdorff phase spaces. In §2, a sufficient condition is given for the property of being closed; when the transformation group is pointwise almost periodic, the condition is equivalent to distal. The problem of whether the proximal relation being closed is preserved by a transformation group homomorphism is considered in §3. In the pointwise almost periodic case, it is shown that the proximal relation itself is actually preserved, and, a fortiori, the closure is preserved. In the general case, a sufficient condition is given for the closure to be preserved.

Let (Z, T) be a transformation group with a uniform phase space Z. If $x, y \in Z$, then x and y are *proximal* if for every index α of Z, there exists $t \in T$ such that $(xt, yt) \in \alpha$. The *proximal relation*, denoted by $P(Z)$, is the set of all pairs of proximal points. It is known [3] that $P(Z)$ is reflexive symmetric and T-invariant but in general is not an equivalence relation or closed. If $z \in Z$, then $zP(Z)$ denotes the set $[w \mid (z, w) \in P(Z)]$. The transformation group (Z, T) is *distal* if every pair of distinct points is not proximal.

Let (W, T) be another transformation group. A *transformation group homomorphism from* (Z, T) *to* (W, T) is a continuous map $f: Z \to W$ such that $f\pi^t = \pi^t f$, for all $t \in T$ [3]. If f is onto, we write $f: (Z, T) \underset{\rightarrow}{\simeq} (W, T)$. We denote the set of all transformation group homomorphisms from (Z, T) to itself by $\operatorname{End}(Z, T)$.

Let Z now be a compact Hausdorff space. The *enveloping semigroup*, denoted by $E(Z)$, is defined to be $[\pi^t \mid t \in T]^- \subset Z^Z$, providing Z^Z with its product topology. As a general reference, see [2]. The class of minimal right ideals will be denoted by $\mathfrak{M}_{E(Z)}$, and the set of idempotents in $\bigcup \mathfrak{M}_{E(Z)}$ will be denoted by $I(Z)$.

Throughout this paper, (X, T) and (Y, T) will denote transformation groups with compact Hausdorff phase spaces X and Y. We will utilize right-handed functional notation. As a general reference, see [4].

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2. **On the proximal relation being closed.** The following lemma will be utilized throughout the paper.

(2.1) LEMMA. *Suppose that there exist* $p \in M \in \mathfrak{M}_{E(X)}$ *such that* p *is continuous. Then the following statements are equivalent:*

(1) $P(X)$ *is an equivalence relation.*

(2) $P(X)$ *is closed.*

PROOF. That (2) implies (1) follows from [**1**, Corollary 1]. We show that (1) implies (2). Since $P(X)$ is an equivalence relation, $\mathfrak{M}_{E(X)} = [M]$ [**2**, Theorem 2]. Let $((x_n, y_n))$ be a net in $P(X)$ such that $(x_n, y_n) \to (x, y)$. By [**2**, Remark 6], for all n and for all $q \in M$, $x_n q = y_n q$. Let p be a continuous element of M. Since for all n, $x_n p = y_n p$ and $(x_n p, y_n p) \to (xp, yp)$, it follows that $xp = yp$. By [**3**, Lemma 4], $(x, y) \in P(X)$. The result follows.

(2.2) THEOREM. *Suppose that there exist* $p \in M \in \mathfrak{M}_{E(X)}$ *such that* p *is continuous and for all* $N \in \mathfrak{M}_{E(X)}$, $pN \cap Np \neq \emptyset$. *Then* $P(X)$ *is closed.*

PROOF. Choose $p \in M \in \mathfrak{M}_{E(X)}$ satisfying the hypotheses. By (2.1) and [**2**, Theorem 2], we need only show that $\mathfrak{M}_{E(X)} = [M]$.

Let $N \in \mathfrak{M}_{E(X)}$. Since $pN \in \mathfrak{M}_{E(X)}$ [**2**, Remark 3] and $pN \subset MN \subset M$, then $pN = M$. Thus, $Np \cap M \neq \emptyset$. But $Np \subset NM \subset N$. It follows that $M \cap N \neq \emptyset$ and $M = N$. The proof is completed.

Note that we have shown that if $p \in M \in \mathfrak{M}_{E(X)}$ and for all $N \in \mathfrak{M}_{E(X)}$, $pN \cap Np \neq \emptyset$, then $P(X)$ is an equivalence relation. The converse is clear.

(2.3) COROLLARY. *Suppose that there exists* $M \in \mathfrak{M}_{E(X)}$ *such that* $M \cap \mathrm{End}(X, T)$ *is nonvacuous. Then* $P(X)$ *is closed.*

PROOF. Let $p \in M \cap \mathrm{End}(X, T)$. Since p is continuous and for all $t \in T$, $p\pi^t = \pi^t p$, it follows that for all $q \in E(X)$, $pq = qp$. Now use (2.2).

(2.4) COROLLARY. *Suppose that* T *is abelian, and there exist* $p \in M \in \mathfrak{M}_{E(X)}$ *such that* p *is continuous. Then* $P(X)$ *is closed.*

PROOF. Since T is abelian, if $p \in E(X)$ and p is continuous, then $p \in \mathrm{End}(X, T)$.

As an example of (2.4), consider the sliding circle.

Now let $p \in M \in \mathfrak{M}_{E(X)}$ such that $pM = Mp$. If $I(M)$ denotes the set of idempotents in M, it is known that $(Mu \mid u \in I(M))$ is a partition of M into subgroups. If $u \in I(M)$ and $p \in Mu$, it follows that $M = Mu$ and M is a group under composition. If, in addition, for all $N \in \mathfrak{M}_{E(X)}$, $pN = Np$, then $\mathfrak{M}_{E(X)} = [M]$ and $I(X)$ is a singleton.

When (X, T) is a pointwise almost periodic transformation group, we can modify (2.2) as follows:

(2.5) THEOREM. *Let (X, T) be a pointwise almost periodic transformation group. Then the following statements are equivalent:*

(1) *There exist $p \in M \in \mathfrak{M}_{E(X)}$ such that p is continuous and for all $N \in \mathfrak{M}_{E(X)}$, $pN = Np$.*

(2) *There exist $p \in M \in \mathfrak{M}_{E(X)}$ and $N \in \mathfrak{M}_{E(X)}$ such that $pN = Np$.*

(3) *(X, T) is a distal transformation group.*

PROOF. It is clear that (1) implies (2). We show that (2) implies (3). Let $p \in M \in \mathfrak{M}_{E(X)}$ and $N \in \mathfrak{M}_{E(X)}$ such that $pN = Np$. Since $pN \subset ME \subset M$ and $Np \subset N$, then $M \cap N \neq \emptyset$ and $M = N$. Thus $pM = Mp$. It follows that M is a group and has a unique idempotent.

Let $N \in \mathfrak{M}_{E(X)}$. Since there is a set isomorphism between the idempotents of M and N [2, Proposition 1], N has a unique idempotent.

Let $x \in X$. Then $xP(X) = xI(X)$. Let $N \in \mathfrak{M}_{E(X)}$ and let $I(N)$ denote the idempotents in N. By [2, Theorem 1], $x \in xI(N)$. Thus, $[x] = xI(N)$. Since $I(X) = \bigcup_{N \in \mathfrak{M}_{E(X)}} I(N)$, then $xI(X) = [x]$. Thus, $xP(X) = [x]$, and (X, T) is a distal transformation group.

We show that (3) implies (1). If (X, T) is a distal transformation group, it is known that $\mathfrak{M}_{E(X)} = [E(X)]$. Thus, $\mathrm{id} = \pi^e \in E(X)$ and id satisfies (1). The proof is completed.

By using the comment in (2.4), we obtain from (2.5):

(2.6) COROLLARY. *Let (X, T) be a pointwise almost periodic transformation group. Let T be abelian. Then the following statements are equivalent:*

(1) *There exists $p \in M \in \mathfrak{M}_{E(X)}$ such that p is continuous.*

(2) *(X, T) is a distal transformation group.*

3. The preservation of the closure of the proximal relation under a homomorphism.

We shall first consider this problem for the pointwise almost periodic case. The following two lemmas are crucial to the proof.

(3.1) LEMMA. *Let f: $(X, T) \underset{\rightarrow}{\simeq} (Y, T)$, and g: $(E(X), T) \underset{\rightarrow}{\simeq} (E(Y), T)$ be induced by f [2, Lemma 4 (1)]. Then $I(X)g = I(Y)$.*

PROOF. Since $\mathfrak{M}_{E(X)}g = \mathfrak{M}_{E(Y)}$ and g is a semigroup homomorphism, it is clear that $I(X)g \subset I(Y)$. We show that $I(Y) \subset I(X)g$. Let

$v \in I(Y)$. It is direct to verify that vg^{-1} is a nonvacuous compact semigroup. Since $\mathfrak{M}_{E(X)}g = \mathfrak{M}_{E(Y)}$, there exists $M \in \mathfrak{M}_{E(X)}$ such that $v \in Mg$. Then $vg^{-1} \cap M$ is a nonvacuous compact semigroup. There exists an idempotent $u \in vg^{-1} \cap M$ [**3**, Lemma 3]. Then $u \in I(X)$ and $ug = v$. The result follows.

(3.2) LEMMA. *Let* $A(X) = [x \mid x \in X$ *and* T *is almost periodic at* $x]$. *Then for all* $x \in X$, $A(X) \cap xP(X) = xI(X)$.

PROOF. Let $x \in X$. We show that $A(X) \cap xP(X) \subset xI(X)$. Let $y \in A(X) \cap xP(X)$. Choose $M \in \mathfrak{M}_{E(X)}$ such that for all $p \in M$, $xp = yp$. Let $I(M)$ denote the set of idempotents in M. Since $y \in A(X)$, then $y \in yI(M)$ [**2**, Theorem 1]. Let $u \in I(M)$ such that $y = yu$. Then $y = yu = xu \in xI(M) \subset xI(X)$. We show that $xI(X) \subset A(X) \cap xP(X)$. Let $y \in xI(X)$. There exists $u \in I(X)$ such that $y = xu$. Then $yu = xu^2 = xu = y$. It follows that $y \in xP(X)$ and, by [**2**, Theorem 1], $y \in A(X)$. The proof is completed.

We now prove the main result of this section.

(3.3) THEOREM. *Let* $f\colon (X, T) \overset{\sim}{\to} (Y, T)$. *Let* $\hat{f}\colon X \times X \to Y \times Y$ *be defined by* $(x, y)\hat{f} = (xf, yf)$. *Let* (Y, T) *be a pointwise almost periodic transformation group. Then* $P(X)\hat{f} = P(Y)$.

PROOF. It is direct to verify that we need only show that for all $x \in X$, $xP(X)f = (xf)P(Y)$. Let $x \in X$. By [**3**, Lemma 2], $xP(X)f \subset (xf)P(Y)$. Let $A(X)$ and g be defined as in (3.2) and (3.1) respectively. By (3.2), $A(X) \cap xP(X) = xI(X)$. Then $xP(X)f \supset xI(X)f = (xf)(I(X)g) = (xf)I(Y)$ by (3.1). Since (Y, T) is a pointwise almost periodic transformation group, $(xf)I(Y) = (xf)P(Y)$. Thus, $xP(X)f \supset (xf)P(Y)$. The proof is completed.

The conclusion of (3.3) is not generally true; see [**3**, Example 2].

Since X and Y are compact Hausdorff spaces and $\hat{f}$ is continuous, we obtain:

(3.4) COROLLARY. *Let* $f\colon (X, T) \overset{\sim}{\to} (Y, T)$. *Let* (Y, T) *be a pointwise almost periodic transformation group. Suppose that* $P(X)$ *is closed. Then* $P(Y)$ *is closed.*

As another application of (3.2) to the proximal relation, we have the following generalization of [**1**, Theorem 4].

(3.5) THEOREM. *Let* $x \in X$. *Let* $M \in \mathfrak{M}_{E(X)}$. *Let* $I(M)$ *and* $A(X)$ *be defined as in* (3.2). *Then*

(1) $A(X) \cap xP(X) = xI(X)$,

(2) *For all* $y, z \in xI(M)$, $(y, z) \in P(X)$. *If* $y \in A(X)$ *and for all* $z \in xI(M)$, $(y, z) \in P(X)$, *then* $y \in xI(M)$.

(3) *Let* $q \in M$ *such that for all* $x \in X$, $(x, xq) \in P(X)$. *Then* $q \in I(M)$.

PROOF. (1) is just (3.2). The proof of (2) proceeds as in [**1**, Theorem 4] using (1), since $y \in A(X)$. To show (3), note that $xq \in A(X)$. By using (2) to show that there exists $v \in I(M)$ such that $xq = xv$, the proof in [**1**, Theorem 4] holds. The result follows.

When (Y, T) is not a pointwise almost periodic transformation group, it is not known whether the conclusion of (3.4) is always valid. We now give a condition which does yield the conclusion without additional hypotheses.

The following lemma is of some independent interest.

(3.6) LEMMA. *Let* $f: (X, T) \underset{\to}{\simeq} (Y, T)$. *Let* $g: (E(X), T) \underset{\to}{\simeq} (E(Y), T)$ *be induced by* f. *Let* $p \in E(X)$ *such that* p *is continuous. Then* pg *is continuous.*

PROOF. Let $y \in Y$. Let $\mathfrak{F}$ be an ultra-filter on Y such that $\mathfrak{F} \to y$. By [**3**, Lemma 7], it is sufficient to show that $\mathfrak{F}(pg) \to y(pg)$.

Now $\mathfrak{F}f^{-1}$ is a filter on X. Let $\mathfrak{G}$ be an ultra-filter on X such that $\mathfrak{G} \supset \mathfrak{F}f^{-1}$. Since X is compact, there exists $x \in X$ such that $\mathfrak{G} \to x$. Moreover, $\mathfrak{G}f \supset \mathfrak{F}f^{-1}f = \mathfrak{F}$, whence $\mathfrak{G}f = \mathfrak{F}$. Thus, $\mathfrak{F} = \mathfrak{G}f \to xf$, and $xf = y$. Since p is continuous, $\mathfrak{F}(pg) = \mathfrak{G}f(pg) = \mathfrak{G}pf \to xpf = (xf)pg = y(pg)$. The result follows.

(3.7) THEOREM. *Let* $f: (X, T) \underset{\to}{\simeq} (Y, T)$. *Suppose that* $P(X)$ *is closed. Let* $\mathfrak{M}_{E(X)} = [M]$. *Suppose further that there exists* $p \in M$ *such that* p *is continuous. Then* $P(Y)$ *is closed.*

PROOF. Let $g: (E(X), T) \underset{\to}{\simeq} (E(Y), T)$ be induced by f. Choose $p \in M$ such that p is continuous. By (3.6), pg is continuous. Moreover, $pg \in Mg \in \mathfrak{M}_{E(Y)}g = \mathfrak{M}_{E(X)}$. By [**2**, Lemma 7], $P(Y)$ is an equivalence relation. It follows from (2.1) that $P(Y)$ is closed. The proof is completed.

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