WHICH MULTIPLICATIVE SEQUENCES ARE MODIFIED CHERN CLASSES?

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Hirzebruch [5] has defined the notion of a multiplicative sequence: It is a natural transformation of group functors

$$\Lambda\colon\ \tilde{K}_c(\ ) \to MH^{**}(\ ;Z),$$

where $\tilde{K}_c(X)$ is the group of virtual unitary bundles over a finite cell complex X [3], [4], and $MH^{**}(X;Z)$ is the multiplicative group of polynomials having 1 as a constant term.

An example of a multiplicative sequence is the total Chern class:

$$C\colon\ \tilde{K}_c(\ ) \to MH^{**}(\ ;Z).$$

In this note we study the question: for what multiplicative sequence Λ does there exist a natural transformation

$$T\colon\ \tilde{K}_c(\ ) \to \tilde{K}_c(\ )$$

such that $T(\alpha+\beta)=T(\alpha)+T(\beta)$ for all $\alpha, \beta \in \tilde{K}_c(X)$ and the following diagram is commutative:

$$\begin{array}{ccc} \tilde{K}_c(\ ) & \xrightarrow{\ \ \Lambda\ \ } & MH^{**}(\ ;Z) \\ & {\scriptstyle T}\searrow \qquad \nearrow{\scriptstyle C} & \\ & \tilde{K}_c(\ ) & \end{array}$$

We shall call such Λ a *modified Chern class.*

In general, if we consider the canonical epimorphism $\epsilon\colon Z \to Z_p$, we shall say that Λ is a modified Chern class mod p if there exists a T such that $\epsilon_*\Lambda = \epsilon_* C \cdot T$.

1. Statement of results. Note that the set of all multiplicative sequences is a group, where $(\Lambda_1+\Lambda_2)(\alpha)=\Lambda_1(\alpha)\cup\Lambda_2(\alpha)$, $(-\Lambda)(\alpha)=\Lambda(-\alpha)=(\Lambda(\alpha))^{-1}$. This group will be denoted by $M(Z)$ if the coefficients are the integers, and $M(Z_p)$ if we consider mod p coefficients, p a prime.

LEMMA. *The modified Chern classes form a subgroup of $M(Z)$; similarly for $M(Z_p)$.*

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PROOF. Immediate. We denote this subgroup by $CM(Z)$, $CM(Z_p)$, respectively.

THEOREM 1. *$CM(Z)$ is a proper subgroup of $M(Z)$. Furthermore, $CM(Z)$ is free abelian and not finitely generated.*

THEOREM 2. *$CM(Z_p)$ is a proper subgroup of $M(Z_p)$. Furthermore, $CM(Z_p)$ is free abelian on $p-1$ generators.*

Let $G\subset Z[[y]]$ be the multiplicative subgroup of power series in a 2-dimensional variable y with constant term 1. Recall:

THEOREM 3 (HIRZEBRUCH). *There is an isomorphism*

$$h: M(Z) \to G \quad \text{such that } h(\Lambda) = \Lambda(\eta - 1),$$

where η is the canonical line bundle over CP^∞.

We can now describe the generators of $CM(Z)$: let

$$g_n = \prod_{j=0}^{n-1} (1 + (n-j)y)^{c_j}, \quad \text{where,} \quad c_j = (-1)^j\binom{n}{j}.$$

THEOREM 4. *Let $\Lambda_n = h^{-1}(g_n)$. Then $CM(Z)$ is generated by $\Lambda_1, \cdots, \Lambda_n, \cdots$. Furthermore, $\epsilon_*\Lambda_1, \cdots, \epsilon_*\Lambda_{p-1}$ form a set of generators for $CM(Z_p)$.*

Let q be a natural number, L_q the subgroup of $M(Z)$ consisting of all Λ such that $\Lambda(\alpha)=1$ for all $\alpha\in\tilde{K}_c(X)$ for X any complex of dimension $\leqq 2q$. Let $CM(Z; q) = CM(Z)/L_q\cap CM(Z)$; $M(Z; q) = M(Z)/L_q$.

COROLLARY 5. *The cosets of $\Lambda_1, \cdots, \Lambda_q$ determine a set of generators for $CM(Z; q)$.*

COROLLARY 6. *$CM(Z;1)\cong M(Z;1)$, $CM(Z;2)\cong M(Z;2)$; $CM(Z;3)$ is not isomorphic to $M(Z;3)$.*

Finally, we list the initial terms (up to degree 7) of the sequences determining Λ_i:

$$\begin{aligned}
g_1 &= 1 + y,\\
g_2 &= 1 - y^2 + 2y^3 - 3y^4 - 4y^5 - 5y^6 + 6y^7,\\
g_3 &= 1 + 2y^3 - 9y^4 + 30y^5 - 88y^6 + 240y^7,\\
g_4 &= 1 - 6y^4 + 48y^5 - 260y^6 + 1200y^7,\\
g_5 &= 1 + 24y^5 - 300y^6 + 2400y^7,\\
g_6 &= 1 - 120y^6 + 2160y^7,\\
g_7 &= 1 + 720y^7.
\end{aligned}$$

2. **Proofs.** It will be helpful to recall a proof of Theorem 3 using the splitting principle [3]. Given an $\alpha \in \tilde{K}_c(X)$, realize it as a $U(n)$-bundle for some sufficiently big h. There is then a space Y_α, a map $f: Y_\alpha \to X$ such that $f^!(\alpha)$ is a Whitney sum of line bundles and f^* is a monomorphism in cohomology. Furthermore, if α and β are two $U(n)$-bundles over X, then $\alpha \times \beta$ is a $U(2n)$-bundle over $X \times X$, and we can take for the space $Y_{\alpha\times\beta}$ the space $Y_\alpha \times Y_\beta$.

Thus a multiplicative sequence Λ is determined by its value on line bundles, hence by its value on the classifying line bundle—thus Theorem 3.

We note that we can use this argument to study additive natural transformations $T: \tilde{K}_c(\ ) \to \tilde{K}_c(\ )$. Such a natural transformation will be determined, once we know its value on line bundles—we let γ_n be the universal n-plane bundle over $X = BU(n)q$, the classifying space for n-plane bundles over complexes of dimensions $\leqq q$. Here $f: Y_{\gamma_n} \to X$ has the property that both $f^!$ and f^* are monomorphisms.

What values can T take on line bundles? This is answered by

THEOREM 7 (ADAMS [1]). *Given a natural number n,*

$$K_c(CP^n) \cong Z[\mu]/(\mu^{n+1}),$$

where $\mu = \eta - 1$, η the canonical line bundle on CP^n with $C(\eta) = 1 + y$, y the two-dimensional fundamental class.

Thus there are as many natural homomorphisms $T: \tilde{K}_c(\ ) \to \tilde{K}_c(\ )$ as there are polynomials in μ. It remains to determine $C \circ T$ for each such T. To do this it is sufficient to determine $C(\mu^n)$ for each n. Now

$$\mu^n = (\eta - 1)^n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \eta^{n-j},$$

hence

$$C(\mu^n) = \prod_{j=0}^{n} C(\eta^{n-j})(-1)^j \binom{n}{j},$$

but $C(\eta^k) = 1 + k\eta$, hence

$$C(\mu^n) = \prod_{j=0}^{n-1} (1 + (n-j)y)^{c_j}, \qquad c_j = (-1)^j \binom{n}{j},$$

which is precisely the power series g_n in §1. This proves the first part of Theorem 4. The second part follows if we note that if $n = \sum_{i=0}^{n} a_i p^i$, $0 \leqq a_i < p$, then $C(\mu^n) = C(\mu^{a_0})$.

We wish to see what the power series $C(\mu^n)$ look like. If we apply the Chern character [4] to μ^n, we know that its leading term is y^n.

Since the filtration [1] of μ^n is at least n, we know that the Chern classes $c_1, \cdots, c_{n-1}$ of μ^n all vanish, hence

$$y^n = ch_n(\mu^n) = \frac{(-1)^n c_n(n-1)}{n!} = (-1)^n \frac{c_n}{(n-1)!}$$

by Newton's formula. Thus $C(\mu^n) = 1 + (-1)^n(n-1)!\ y^n + \cdots$ higher terms.

This indicates a way of finding out whether a given series

$$Q(y) = 1 + a_1 y + a_2 y^2 + \cdots$$

determines a multiplicative sequence which agrees with a modified Chern class through any finite dimensional k—we do this by stripping off a coefficient at a time.

We notice that since $C(\mu) = 1 + y$, $C(\mu^2) = 1 - y^2 +$ higher terms, that $Q(y)$ always agrees with a modified Chern class in dimensions $\leqq 2$ (these are the first two assertions of Corollary 6). We wish to exhibit a $Q(y)$ which fails to agree with a modified Chern class in dimension 3. The simplest such example is

$$Q(y) = 1 + y^3,$$

for if $Q(y) = \prod_{n=1} C(\mu^n)^{b_n}$, then $b_1 = b_2 = 0$, $b_3 \neq 0$, but then the leading term must have coefficient divisible by 2.

EXAMPLE.

$$
\begin{aligned}
Q(y) &= 1 + y + y^2 + y^3 + 2y^4 + \cdots, \\
Q(y)(1+y)^{-1} &= 1 + y^2 + 2y^4 + \cdots, \\
Q(y)(1+y)^{-1}C(\mu^2)^1 &= 1 + 2y^3 - 2y^4 + \cdots, \\
Q(y)(1+y)^{-1}C(\mu^2)^1 C(\mu^3)^{-1} &= 1 + 7y^4 + \cdots.
\end{aligned}
$$

Therefore $Q(y) = 1 + y + y^2 + y^3 + 2y^4 + \cdots$ agrees with $\Lambda_1^1 + \Lambda_2^{-1} + \Lambda_3^1$ through dimension 3, but disagrees with every modified Chern sequence in dimension 4.

Theorem 1 now follows, because there are no relations on the sequences g_n in $G \subset Z[[y]]$. Theorem 2 is proved by remarking that $G_p \subset Z_p[[y]]$ is not finitely generated.

REMARK. The referee points out that

$$g_n' = 1 + ny$$

are also generators of $CM(Z)$. They correspond to $\eta^n - 1$ whereas the g_n belong to $(\eta - 1)^n$. The generators g_n are used for filtration purposes and are of interest for Corollaries 5 and 6. The g_n' are related to Adams' operations ψ_n.

References

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