CREATIVE AND WEAKLY CREATIVE SEQUENCES OF r.e. SETS

V. D. VUCKOVIC

1. In [1] Cleave introduced the notion of a creative sequence of r.e. (recursively enumerable) sets and proved that all such sequences are r. (recursively) isomorphic and 1-1 universal for the class of all r.e. sequences of r.e. sets. In [2] and [3] Lachlan introduced an alternate definition and proved its equivalency with the definition of Cleave.

A sequence of r.e. sets $E_0, E_1, \ldots$ is called r.e. iff there is an r. function $g$ such that $E_i = w_{g(i)}$ for every $i \in \mathbb{N}$, where

\[(1.1) \quad x \in w_i \iff \forall T_1(i, x, y).\]

Cleave calls a disjoint r.e. sequence $E_0, E_1, \ldots$ of r.e. sets creative if there is a p. (partial) r. function $f$ such that for every disjoint r.e. sequence $w_{h(i)}, i = 0, 1, \ldots$, (with recursive $h$) satisfying $E_i \cap w_{h(i)} = \emptyset$, for all $i$, we have, for every $x \in I(h),$

\[(1.2) \quad f(x) \in \bigcup_{\mu=0}^{\infty} (w_{h(\mu)} \cup E_{\mu}).\]

$I(h)$ is the set of indices of $h$ in the standard enumeration

\[(1.3) \quad \phi_0, \phi_1, \phi_2, \ldots,\]

of all r.p. functions, i.e.,

\[(1.4) \quad \phi_i(x) \simeq U(\mu T_1(i, x, y)).\]

Lachlan, in [2], proceeds as follows. Let first $g$ be recursive and such that

\[\forall T_2(i, n, x, y) \leftrightarrow \forall T_1(g(i, n), x, y).\]

Define the double sequence $W_{i,n}$ of r.e. sets by $W_{i,n} = w_{g(i,n)}$.

After Lachlan, an r.e. sequence $E_0, E_1, \ldots$ of r.e. sets is creative iff there is a recursive $f$ such that for all $i$

\[(1.5) \quad W_{i,f(i)} \cup E_{f(i)} \subseteq \bigcup_{\mu=0}^{\infty} (W_{i,\mu} \cap E_{\mu}).\]

Received by the editors March 1, 1967.
Both Cleave’s and Lachlan’s definition seem to demand very much to be satisfied: (1.1) involves all indices \( x \) of \( h \), and (1.5) all indices \( i \) (which are, in essence, indices of all r.e. sequences).

In this paper we propose a very weak definition of a creative sequence and prove its equivalency with the definition of Cleave (and so with the definition of Lachlan). Moreover, our definition is a direct generalization of the corresponding Smullyan’s definition of a doubly weakly creative pair (Smullyan [4, p. 114]).

2. Obviously, a sequence \( A_0, A_1, \cdots \) of r.e. sets is r.e. iff the predicate \( x \in A_y \) is r.e. Let \( \gamma \) be recursive and such that

\[
V \ T_2(u, \mu, x, y) \leftrightarrow V \ T_1(\gamma(\mu, u), x, y).
\]

For every r.e. predicate \( Q(\mu, x) \) there is an \( e \in \mathbb{N} \) such that \( Q(\mu, x) \leftrightarrow \forall y \ T_2(e, \mu, x, y) \). With \( Q(\mu, x) \leftrightarrow x \in A_\mu \) we conclude: every r.e. sequence of r.e. sets can be represented as a sequence \( w_{\gamma(\mu, e)}(\mu) \) \( \mu = 0, 1, \cdots \) for some \( e \).

By the recursion theorem, for every r.e. predicate \( Q(\mu, z, x, u) \) there is a recursive \( \phi \) such that for all \( i \in \mathbb{N} \),

\[
Q(\mu, i, x, \phi(i)) \leftrightarrow V \ T_2(\phi(i), \mu, x, y).
\]

i.e., by (2.1),

\[
Q(\mu, i, x, \phi(i)) \leftrightarrow V \ T_1(\gamma(\mu, \phi(i)), x, y).
\]

**Lemma 2.1.** Let \( A_0, A_1, \cdots \) be an r.e. sequence of r.e. sets and let \( f \) be any r. function. Then there is an r. function \( \phi \) such that, for every \( i \in \mathbb{N} \),

\[
i \in A_\mu \rightarrow w_{\gamma(\mu, \phi(i))} = \{f(\phi(i))\};
\]

\[
i \notin A_\mu \rightarrow w_{\gamma(\mu, \phi(i))} = \emptyset
\]

**Proof.** (\( \{a\} \) denotes the singleton whose unique element is \( a \); \( \emptyset \) is the empty set.) In (2.3) take

\[
Q(\mu, z, x, u) \leftrightarrow z \in A_\mu \land x = f(u).
\]

From this lemma we obtain immediately.

**Lemma 2.2.** Let \( A_0, A_1, \cdots \) be a disjoint r.e. sequence of r.e. sets. Then there is an r. function \( \phi \) such that, for every \( i \in \mathbb{N} \),

\[
i \in A_\mu \rightarrow w_{\gamma(\mu, \phi(i))} = \{f(\phi(i))\} \text{ and all others } w_{\gamma(\nu, \phi(i))} \text{ are empty for } \nu \neq \mu, \text{ and}
\]

\[
i \notin A_\mu \rightarrow \text{all } w_{\gamma(\mu, \phi(i))} \text{ are empty.}
\]
Definition 2.1. An r.e. sequence \( A_0, A_1, \ldots \) of r.e. sets is meager iff either all \( A_\mu \) are empty or all but one are empty and this one, which is not empty, is a singleton.

Definition 2.2. A disjoint r.e. sequence \( A_0, A_1, \ldots \) of r.e. sets is weakly creative under an r. function \( f \) iff, for all \( i \in \mathbb{N} \) for which the sequence \( w_{\gamma(0,i)}, w_{\gamma(1,i)}, \ldots \) is meager,

(a) in case all \( w_{\gamma(\mu,i)} \) are empty we have

\[
(2.8) \quad f(i) \in \bigcup_{\mu=0}^{\infty} A_\mu;
\]

(b) in case \( w_{\gamma(n_1,i)} \) is not empty and \( w_{\gamma(n_0,i)} \cap A_{n_0} = \emptyset \), we have

\[
(2.9) \quad f(i) \in w_{\gamma(n_0,i)}.
\]

3. We prove some theorems from which will follow the equivalency of the weak creativity and the creativity in the sense of Cleave.

Theorem 3.1. If the sequence \( E = E_0, E_1, \ldots \) is weakly creative then every disjoint r.e. sequence \( A = A_0, A_1, \ldots \) of r.e. sets is reducible to \( E \).

Proof. Let \( E \) be creative under \( f \). By Lemma 2.2 there is an r. function \( \phi \) such that for every sequence \( \Omega_i = w_{\gamma(0,\phi(i))}, w_{\gamma(1,\phi(i))}, \ldots \), we have

\[
(3.1) \quad i \in A_\mu \rightarrow \Omega_i \text{ is meager and } w_{\gamma(\mu,\phi(i))} = \{f(\phi(i))\}, \text{ and}
\]

\[
(3.2) \quad i \in A_\mu \rightarrow \Omega_i \text{ is meager and all } w_{\gamma(\mu,\phi(i))} \text{ are empty}.
\]

We shall prove that \( \psi = f(\phi) \) reduces \( A \) to \( E \).

Suppose first that \( i \in A_\mu \). Then \( w_{\gamma(\mu,\phi(i))} = \{f(\phi(i))\} \) and, therefore,

\[
(3.3) \quad f(\phi(i)) \in w_{\gamma(\mu,\phi(i))}.
\]

If now \( w_{\gamma(\mu,\phi(i))} \cap E_\mu = \emptyset \) we will have, by (2.9), \( f(\phi(i)) \in w_{\gamma(\mu,\phi(i))} \) in contradiction to (3.3). Therefore, \( f(\phi(i)) \in E_\mu \).

To prove the opposite inclusion

\[
(3.4) \quad f(\phi(i)) \in E_\mu \rightarrow i \in A_\mu
\]

suppose, contrary, that there is a \( q \in \mathbb{N} \) such that \( f(\phi(i)) \in E_q \) but \( i \notin A_q \).

Now, if \( i \in \bigcup_{\mu=0}^{\infty} A_\mu \), \( \Omega_i \) consists of empty sets only, and (2.8) gives

\[
\text{f(\phi(i))} \notin \bigcup_{\mu=0}^{\infty} E_\mu - \text{a contradiction. So, there is an } s \in \mathbb{N} \text{ such that } i \in A_s. \text{ By the first part of the proof we obtain } f(\phi(i)) \in E_s. \text{ As } E_s \cap E_q = \emptyset \text{ for } q \neq s, \text{ it follows } s = q.
\]

So we have proved

\[
(3.5) \quad i \in A_\mu \rightarrow \psi(i) \in E_\mu
\]

i.e. that \( A \) is r. reducible to \( E \).
Theorem 3.2. If the creative sequence \( A = A_0, A_1, \ldots \), is reducible to \( B = B_0, B_1, \ldots \), then \( B \) is a creative sequence.

Proof. Let \( A \) be creative under \( \rho \). Therefore, for every disjoint r.e. sequence \( w_h(\mu), \mu = 0, 1, \ldots \), satisfying \( A_\mu \cap w_h(\mu) = \emptyset \) for all \( \mu \), if \( x \in I(h) \) then
\[
\rho(x) \notin \bigcup_{\mu=0}^{\infty} (w_h(\mu) \cup A_\mu).
\]

If \( f \) reduces \( A \) to \( B \) then
\[
A_\mu = f^{-1}(B_\mu), \quad \mu = 0, 1, \ldots.
\]

Denote by \( \psi \) the r. function such that for all \( x \in \mathbb{N} \)
\[
w_{\psi(x)} = f^{-1}(w_x).
\]

There is a recursive function \( \phi \) such that if \( x \in I(F) \) then \( \phi(x) \in I(\psi(F)) \) (the operation of composition being effective). We shall prove that \( B \) is creative under \( \chi = f(\rho(\phi)) \).

Let \( w_{k(0)}, w_{k(1)}, \ldots \), be any disjoint r.e. sequence of r.e. sets such that
\[
w_{k(\mu)} \cap B_\mu = \emptyset \quad \text{for all} \ \mu,
\]
and let \( x \) be an index of the r. function \( k \). We have to prove
\[
\chi(x) \notin \bigcup_{\mu=0}^{\infty} (w_{k(\mu)} \cup B_\mu).
\]

By (3.9), using (3.7) and (3.8), we have
\[
A_\mu \cap w_{\psi(k(\mu))} = \emptyset, \quad \text{for all} \ \mu.
\]
As \( A \) is creative and as \( \phi(x) \in I(\psi(k)) \), we get by (3.6)
\[
\rho(\phi(x)) \notin \bigcup_{\mu=0}^{\infty} (A_\mu \cup w_{\psi(k(\mu))}).
\]

From (3.7), (3.8) and (3.12) follows now (3.10).

Corollary 3.2.1. If a sequence \( A \) is weakly creative it is creative.

Proof. Every creative sequence is reducible to \( A \) by Theorem 3.1. By Theorem 3.2, \( A \) is creative.

Theorem 3.3. If a weakly creative sequence \( A = A_0, A_1, \ldots \), is 1-1 reducible to \( B = B_0, B_1, \ldots \), then \( B \) is a weakly creative sequence.

Proof. Let \( A \) be weakly creative under \( \phi \) and let the 1-1 r. function

\[
\rho(x) \notin \bigcup_{\mu=0}^{\infty} (w_h(\mu) \cup A_\mu).
\]
f reduce A to B. There is a recursive ψ such that, for all x ∈ N,
\[ w_\gamma(\mu, \psi(z)) = f^{-1}(w_\gamma(\mu, z)), \quad \mu = 0, 1, \ldots, \]

(Take in (2.3) Q(μ, z, x, u) ⇔ x ∈ f^{-1}(w_\gamma(\mu, u)) \land z = z.)

Let \( w_\gamma(0, i), w_\gamma(1, i), \ldots \), be a meager sequence. Then \( w_\gamma(0, \psi(i)) \), \( w_\gamma(1, \psi(i)) \), \( \ldots \), is meager too.

Suppose first that \( w_\gamma(n_0, i) \neq \emptyset \) and that \( w_\gamma(n_0, i) \cap \mathcal{B}_{n_0} = \emptyset \). Then \( w_\gamma(n_0, \psi(i)) \cap \mathcal{A}_{n_0} = \emptyset \) and, as \( f \) is 1-1, \( w_\gamma(n_0, \psi(i)) \) is a singleton. Then \( \phi(\psi(i)) \notin w_\gamma(n_0, \psi(i)) \) and, as
\[ y \in w_\gamma(n_0, \psi(i)) \iff f(y) \in w_\gamma(n_0, i), \]
we obtain \( f(\phi(\psi(i))) \notin w_\gamma(n_0, i) \).

If all \( w_\gamma(\mu, i) \) are empty, from \( \phi(\psi(i)) \notin \bigcup_{\mu = 0}^{\infty} A_\mu \) we obtain \( f(\phi(\psi(i))) \notin \bigcup_{\mu = 0}^{\infty} B_\mu \).

This proves that B is weakly creative under \( f(\phi(\psi)) \).

**Corollary 3.3.1.** Every creative sequence is weakly creative.

**Proof.** By part (3) of Corollary 4 of Cleave's paper [1], every weakly creative sequence is 1-1 r.e. reducible to every creative sequence. By Theorem 3.3 follows the statement.

Corollaries 3.2.1 and 3.3.1 give

**Theorem 3.4.** A sequence is weakly creative iff it is creative.

We point out that using the Definition 3.4 of the paper [2] of Lachlan one can give a definition of M-creativity (akin to Lachlan's definition of M-coproductivity) which is similar to our definition of weak creativity, but unnecessarily complicated. Namely, starting from the sequence \( A = A_0, A_1, \ldots \), Lachlan constructs the sequence \( A^* = A_0^*, A_1^*, \ldots \), where
\[ A_\mu^* = A_\mu \quad \text{if } A_\mu \text{ is a singleton}, \]
\[ = \emptyset \quad \text{otherwise}. \]

With this definition, A will be called M-creative under f iff A is a r.e. sequence of r.e. sets and iff for all i
\[ \bigcup_{\mu = 0}^{\infty} (W_{i, \mu} \cap A_\mu^*) = \emptyset \rightarrow \{ f(i) \text{ is defined and } W_{i, f(i)} = A_i = \emptyset \}. \]

(\( W_{i, f(i)} \) is as in §1.) As M-creativity is equivalent with creativity it is equivalent with weak creativity.

On the ground of the Theorem 3.4 one can propose the following
definition of creativity, which we shall call $S$-creativity:

A disjoint r.e. sequence $A = A_0, A_1, \ldots$, of r.e. sets is $S$-creative under a recursive $f$ iff for every disjoint sequence $w_{\gamma(0,i)}, w_{\gamma(1,i)}, \ldots$, for which $A_\mu \cap w_{\gamma(\mu,i)} = \emptyset$ for all $\mu$, we have

$$f(i) \in \bigcup_{\mu=0}^{\infty} (A_\mu \cup w_{\gamma(\mu,i)}).$$

It is not difficult to prove that a sequence is $S$-creative iff it is creative. The implication “$S$-creative $\rightarrow$ creative” is trivial. The converse implication is obtained through a theorem, similar to Theorem 3.3.

References