CREATIVE AND WEAKLY CREATIVE SEQUENCES OF r.e. SETS

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1. In [1] Cleave introduced the notion of a creative sequence of r.e. (recursively enumerable) sets and proved that all such sequences are r. (recursively) isomorphic and 1-1 universal for the class of all r.e. sequences of r.e. sets. In [2] and [3] Lachlan introduced an alternate definition and proved its equivalency with the definition of Cleave.

A sequence of r.e. sets $E_0, E_1, \cdots$ is called r.e. iff there is an r. function g such that $E_i = w_{g(i)}$ for every $i \in N$, where

$$(1.1)\qquad x \in w_i \leftrightarrow \bigvee_y T_1(i, x, y).$$

Cleave calls a disjoint r.e. sequence $E_0, E_1, \cdots$ of r.e. sets *creative* if there is a p. (partial) r. function f such that for every disjoint r.e. sequence $w_{h(i)}$, $i = 0, 1, \cdots$, (with recursive h) satisfying $E_i \cap w_{h(i)} = \emptyset$, for all i, we have, for every $x \in I(h)$,

$$(1.2)\qquad f(x) \notin \bigcup_{\mu=0}^{\infty} (w_{h(\mu)} \cup E_\mu).$$

$I(h)$ is the set of indices of h in the standard enumeration

$$(1.3)\qquad \phi_0, \phi_1, \phi_2, \cdots,$$

of all r.p. functions, i.e.,

$$(1.4)\qquad \phi_i(x) \simeq U(\mu_y T_1(i, x, y)).$$

Lachlan, in [2], proceeds as follows. Let first g be recursive and such that

$$\bigvee_y T_2(i, n, x, y) \leftrightarrow \bigvee_y T_1(g(i, n), x, y).$$

Define the double sequence $W_{i,n}$ of r.e. sets by $W_{i,n} = w_{g(i,n)}$.

After Lachlan, an r.e. sequence $E_0, E_1, \cdots$ of r.e. sets is creative iff there is a recursive f such that for all i

$$(1.5)\qquad W_{i,f(i)} \cup E_{f(i)} \subset \bigcup_{\mu=0}^{\infty} (W_{i,\mu} \cap E_\mu).$$

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Both Cleave's and Lachlan's definition seem to demand very much to be satisfied: (1.1) involves all indices x of h, and (1.5) all indices i (which are, in essence, indices of all r.e. sequences).

In this paper we propose a very weak definition of a creative sequence and prove its equivalency with the definition of Cleave (and so with the definition of Lachlan). Moreover, our definition is a direct generalization of the corresponding Smullyan's definition of a doubly weakly creative pair (Smullyan [4, p. 114]).

2. Obviously, a sequence $A_0, A_1, \cdots$ of r.e. sets is r.e. iff the predicate $x \in A_y$ is r.e. Let γ be recursive and such that

$$\bigvee_y T_2(u, \mu, x, y) \leftrightarrow \bigvee_y T_1(\gamma(\mu, u), x, y). \tag{2.1}$$

For every r.e. predicate $Q(\mu, x)$ there is an $e \in N$ such that $Q(\mu, x) \leftrightarrow \bigvee_y T_2(e, \mu, x, y)$. With $Q(\mu, x) \leftrightarrow x \in A_\mu$ we conclude: every r.e. sequence of r.e. sets can be represented as a sequence $w_{\gamma(\mu,e)}$ $\mu = 0, 1, \cdots$ for some e.

By the recursion theorem, for every r.e. predicate $Q(\mu, z, x, u)$ there is a recursive ϕ such that for all $i \in N$,

$$Q(\mu, i, x, \phi(i)) \leftrightarrow \bigvee_y T_2(\phi(i), \mu, x, y) \tag{2.2}$$

i.e., by (2.1),

$$Q(\mu, i, x, \phi(i)) \leftrightarrow \bigvee_y T_1(\gamma(\mu, \phi(i)), x, y). \tag{2.3}$$

LEMMA 2.1. *Let $A_0, A_1, \cdots$ be an r.e. sequence of r.e. sets and let f be any r. function. Then there is an r. function ϕ such that, for every $i \in N$,*

$$i \in A_\mu \to w_{\gamma(\mu,\phi(i))} = \{f(\phi(i))\}; \tag{2.4}$$

$$i \notin A_\mu \to w_{\gamma(\mu,\phi(i))} = \varnothing \tag{2.5}$$

PROOF. ($\{a\}$ denotes the singleton whose unique element is a; $\varnothing$ is the empty set.) In (2.3) take

$$Q(\mu, z, x, u) \leftrightarrow z \in A_\mu \wedge x = f(u).$$

From this lemma we obtain immediately.

LEMMA 2.2. *Let $A_0, A_1, \cdots$ be a disjoint r.e. sequence of r.e. sets. Then there is an r. function ϕ such that, for every $i \in N$,*

(2.6) *$i \in A_\mu \to w_{\gamma(\mu,\phi(i))} = \{f(\phi(i))\}$ and all others $w_{\gamma(\nu,\phi(i))}$ are empty for $\nu \neq \mu$, and*

(2.7) *$i \notin A_\mu \to$ all $w_{\gamma(\mu,\phi(i))}$ are empty.*

DEFINITION 2.1. An r.e. sequence $A_0, A_1, \cdots$ of r.e. sets is *meager* iff either all A_μ are empty or all but one are empty and this one, which is not empty, is a singleton.

DEFINITION 2.2. A disjoint r.e. sequence $A_0, A_1, \cdots$ of r.e. sets is *weakly creative* under an r. function f iff, for all $i \in N$ for which the sequence $w_{\gamma(0,i)}, w_{\gamma(1,i)}, \cdots$ is meager,

(a) in case all $w_{\gamma(\mu,i)}$ are empty we have

$$f(i) \notin \bigcup_{\mu=0}^{\infty} A_\mu; \tag{2.8}$$

(b) in case $w_{\gamma(n_1,i)}$ is not empty and $w_{\gamma(n_0,i)} \cap A_{n_0} = \varnothing$, we have

$$f(i) \notin w_{\gamma(n_0,i)}. \tag{2.9}$$

3. We prove some theorems from which will follow the equivalency of the weak creativity and the creativity in the sense of Cleave.

THEOREM 3.1. *If the sequence $E = E_0, E_1, \cdots$ is weakly creative then every disjoint r.e. sequence $A = A_0, A_1, \cdots$ of r.e. sets is reducible to E.*

PROOF. Let E be creative under f. By Lemma 2.2 there is an r. function ϕ such that for every sequence $\Omega_i = w_{\gamma(0,\phi(i))}, w_{\gamma(1,\phi(i))}, \cdots$, we have

(3.1) $i \in A_\mu \rightarrow \Omega_i$ is meager and $w_{\gamma(\mu,\phi(i))} = \{f(\phi(i))\}$, and

(3.2) $i \notin A_\mu \rightarrow \Omega_i$ is meager and all $w_{\gamma(\mu,\phi(i))}$ are empty.

We shall prove that $\psi = f(\phi)$ reduces A to E.

Suppose first that $i \in A_\mu$. Then $w_{\gamma(\mu,\phi(i))} = \{f(\phi(i))\}$ and, therefore,

$$f(\phi(i)) \in w_{\gamma(\mu,\phi(i))}. \tag{3.3}$$

If now $w_{\gamma(\mu,\phi(i))} \cap E_\mu = \varnothing$ we will have, by (2.9), $f(\phi(i)) \notin w_{\gamma(\mu,\phi(i))}$ in contradiction to (3.3). Therefore, $f(\phi(i)) \in E_\mu$.

To prove the opposite inclusion

$$f(\phi(i)) \in E_\mu \rightarrow i \in A_\mu \tag{3.4}$$

suppose, contrary, that there is a $q \in N$ such that $f(\phi(i)) \in E_q$ but $i \notin A_q$.

Now, if $i \notin \bigcup_{\mu=0}^{\infty} A_\mu$, Ω_i consists of empty sets only, and (2.8) gives $f(\phi(i)) \notin \bigcup_{\mu=0}^{\infty} E_\mu$—a contradiction. So, there is an $s \in N$ such that $i \in A_s$. By the first part of the proof we obtain $f(\phi(i)) \in E_s$. As $E_s \cap E_q = \varnothing$ for $q \neq s$, it follows $s = q$.

So we have proved

$$i \in A_\mu \leftrightarrow \psi(i) \in E_\mu \tag{3.5}$$

i.e. that A is r. reducible to E.

THEOREM 3.2. *If the creative sequence* $A = A_0, A_1, \cdots,$ *is reducible to* $B = B_0, B_1, \cdots,$ *then* B *is a creative sequence.*

PROOF. Let A be creative under p. Therefore, for every disjoint r.e. sequence $w_{h(\mu)}$, $\mu = 0, 1, \cdots,$ satisfying $A_\mu \cap w_{h(\mu)} = \varnothing$ for all μ, if $x \in I(h)$ then

$$p(x) \notin \bigcup_{\mu=0}^{\infty} (w_{h(\mu)} \cup A_\mu). \tag{3.6}$$

If f reduces A to B then

$$A_\mu = f^{-1}(B_\mu), \qquad \mu = 0, 1, \cdots. \tag{3.7}$$

Denote by ψ the r. function such that for all $x \in N$

$$w_{\psi(x)} = f^{-1}(w_x). \tag{3.8}$$

There is a recursive function ϕ such that if $x \in I(F)$ then $\phi(x) \in I(\psi(F))$ (the operation of composition being effective). We shall prove that B is creative under $\chi = f(p(\phi))$.

Let $w_{k(0)}, w_{k(1)}, \cdots,$ be any disjoint r.e. sequence of r.e. sets such that

$$w_{k(\mu)} \cap B_\mu = \varnothing \qquad \text{for all } \mu, \tag{3.9}$$

and let x be an index of the r. function k. We have to prove

$$\chi(x) \notin \bigcup_{\mu=0}^{\infty} (w_{k(\mu)} \cup B_\mu). \tag{3.10}$$

By (3.9), using (3.7) and (3.8), we have

$$A_\mu \cap w_{\psi(k(\mu))} = \varnothing, \qquad \text{for all } \mu. \tag{3.11}$$

As A is creative and as $\phi(x) \in I(\psi(k))$, we get by (3.6)

$$p(\phi(x)) \notin \bigcup_{\mu=0}^{\infty} (A_\mu \cup w_{\psi(k(\mu))}). \tag{3.12}$$

From (3.7), (3.8) and (3.12) follows now (3.10).

COROLLARY 3.2.1. *If a sequence* A *is weakly creative it is creative.*

PROOF. Every creative sequence is reducible to A by Theorem 3.1. By Theorem 3.2, A is creative.

THEOREM 3.3. *If a weakly creative sequence* $A = A_0, A_1, \cdots,$ *is* 1-1 *reducible to* $B = B_0, B_1, \cdots,$ *then* B *is a weakly creative sequence.*

PROOF. Let A be weakly creative under ϕ and let the 1-1 r. function

f reduce A to B. There is a recursive ψ such that, for all $x \in N$,

$$w_{\gamma(\mu,\psi(x))} = f^{-1}(w_{\gamma(\mu,x)}), \qquad \mu = 0, 1, \cdots,$$

(Take in (2.3) $Q(\mu, z, x, u) \leftrightarrow x \in f^{-1}(w_{\gamma(\mu,u)}) \wedge z = z$.)

Let $w_{\gamma(0,i)}, w_{\gamma(1,i)}, \cdots$, be a meager sequence. Then $w_{\gamma(0,\psi(i))}, w_{\gamma(1,\psi(i))}, \cdots$, is meager too.

Suppose first that $w_{\gamma(n_0,i)} \neq \emptyset$ and that $w_{\gamma(n_0,i)} \cap B_{n_0} = \emptyset$. Then $w_{\gamma(n_0,\psi(i))} \cap A_{n_0} = \emptyset$ and, as f is 1-1, $w_{\gamma(n_0,\psi(i))}$ is a singleton. Then $\phi(\psi(i)) \notin w_{\gamma(n_0,\psi(i))}$ and, as

$$y \in w_{\gamma(n_0,\psi(i))} \leftrightarrow f(y) \in w_{\gamma(n_0,i)},$$

we obtain $f(\phi(\psi(i))) \notin w_{\gamma(n_0,i)}$.

If all $w_{\gamma(\mu,i)}$ are empty, from $\phi(\psi(i)) \notin \bigcup_{\mu=0}^{\infty} A_\mu$ we obtain $f(\phi(\psi(i))) \notin \bigcup_{\mu=0}^{\infty} B_\mu$.

This proves that B is weakly creative under $f(\phi(\psi))$.

COROLLARY 3.3.1. *Every creative sequence is weakly creative.*

PROOF. By part (3) of Corollary 4 of Cleave's paper [1], every weakly creative sequence is 1-1 r. reducible to every creative sequence. By Theorem 3.3 follows the statement.

Corollaries 3.2.1 and 3.3.1 give

THEOREM 3.4. *A sequence is weakly creative iff it is creative.*

We point out that using the Definition 3.4 of the paper [2] of Lachlan one can give a definition of M-creativity (akin to Lachlan's definition of M-coproductivity) which is similar to our definition of weak creativity, but unnecessarily complicated. Namely, starting from the sequence $A = A_0, A_1, \cdots$, Lachlan constructs the sequence $A^* = A_0^*, A_1^*, \cdots$, where

$$\begin{aligned} A_\mu^* &= A_\mu && \text{if } A_\mu \text{ is a singleton,} \\ &= \emptyset && \text{otherwise.} \end{aligned}$$

With this definition, A will be called M-creative under f iff A is a r.e. sequence of r.e. sets and iff for all i

$$\bigcup_{\mu=0}^{\infty} (W_{i,\mu} \cap A_\mu^*) = \emptyset \rightarrow \{f(i) \text{ is defined and } W_{i,f(i)} = A_i = \emptyset\}.$$

($W_{i,j}$ is as in §1.) As M-creativity is equivalent with creativity it is equivalent with weak creativity.

On the ground of the Theorem 3.4 one can propose the following definition of creativity, which we shall call S-creativity:

A disjoint r.e. sequence $A = A_0, A_1, \cdots$, of r.e. sets is S-creative under a recursive f iff for every disjoint sequence $w_{\gamma(0,i)}, w_{\gamma(1,i)}, \cdots$, for which $A_\mu \cap w_{\gamma(\mu,i)} = \emptyset$ for all μ, we have

$$f(i) \notin \bigcup_{\mu=0}^{\infty} (A_\mu \cup w_{\gamma(\mu,i)}).$$

It is not difficult to prove that a sequence is S-creative iff it is creative. The implication "S-creative$\rightarrow$creative" is trivial. The converse implication is obtained through a theorem, similar to Theorem 3.3.

References

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