THE WEAK RADICAL OF A RING

KWANGIL KOH AND A. C. MEWBORN

1. Introduction. In [2] a *weakly transitive* ring of linear transformations is defined as follows:

DEFINITION A. Let V be a left vector space over a division ring D and let R be a ring of linear transformations of V. Consider V as right R-module. Then R is *weakly transitive* provided there is a right order K in D and a (K, R)-submodule M of V such that M is *uniform* as R-module, $DM=V$, and such that if $\{m_i\}_{i=1}^n$ is a finite D-linearly independent subset of M and if $\{y_i\}_{i=1}^n$ is a sequence from M, then there exists $r\in R$, $k\in K$, $k\neq 0$, such that $m_ir=ky_i$, $1\leqq i\leqq n$.

By a *weakly transitive representation* of a ring R we mean a homomorphism of R onto a weakly transitive ring.

DEFINITION 1.1. If R is a ring the *weak radical*, $W(R)$, of R is the intersection of the set of all ideals which are kernels of weakly transitive representations. If R has no weakly transitive representation, then $W(R)=R$.

Of importance in studying weakly transitive rings is the notion of an *almost maximal* right ideal, which is also defined in [2].

DEFINITION B. If I is a proper right ideal of a ring R then I is *almost maximal* provided that

(1) if J_1 and J_2 are right ideals of R and $J_1\cap J_2=I$, then $J_1=I$ or $J_2=I$, i.e. I is irreducible,

(2) if $a\in R$ and $[I:a]$[1] $\supset I$, then $a\in I$,

(3) if J is a right ideal of R, $J\supset I$, then $N(I)\cap J\supset I$, where $N(I)=\{r\in R: rI\subseteq I\}$, and if $a\in R$ such that $[J:a]\supseteq I$ then $[J:a]\supset I$.

It was shown in [2] that if I is an almost maximal right ideal of R, then $M=R/I$ is a uniform right R-module with centralizer $K=N(I)/I$. M has extended centralizer D [see **2**, p. 67] in which K is a right order, and M can be extended to a (D, R)-module V such that $DM=V$ and such that R induces a weakly transitive ring of linear transformations of the D-space V.

In this paper we observe that $W(R/W(R))=(0)$ and prove that $W(R)$ is the intersection of the set of almost maximal right ideals of R. From the fact that a (two-sided) ideal of a weakly transitive ring is also weakly transitive we deduce that if S is an ideal of R then $W(S)=W(R)\cap S$. Finally, if n is a positive integer then $W(R_n)=W(R)_n$.

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[1] By $[I: a]$, we mean the set $\{r\in R: ar\in I\}$.

It is interesting to note that in severel instances the usual proofs of the analogues of our theorems in the Jacobson structure theory (See, for example, [1, p. 10]) make very strong use of the notion of quasi-regularity. Since in the weak radical theory we have no analogue of this notion the proofs which we give suggest proofs of the classical theorems which do not depend on quasi-regularity. The authors have been unable to devise a proof of the equality of the left and right radicals without using quasi-regularity and this problem for the weak radical remains open.

From the definition of $W(R)$ it follows that if $W(R)=(0)$, then R is a subdirect sum of weakly transitive rings. It is likely that this hypothesis of "weak semisimplicity" will serve in many cases where a ring is assumed to be semisimple.

2. The proof of the following theorem is straightforward and will be omitted.

THEOREM 2.1. *If R is a ring then $W(R/W(R))=(0)$.*

LEMMA 2.2. *If I is an almost maximal right ideal of a ring R and $M=R/I$, then for each $m\in M$, $m\neq 0$, the right ideal $m^\gamma=\{r\in R: mr=0\}$ is almost maximal.*

PROOF. mR is a submodule of M and hence is uniform. Since $mR\cong R/m^\gamma$, m^γ is an irreducible right ideal. Suppose $a\in R$ and $[m^\gamma:a]=(ma)^\gamma\supset m^\gamma$. By Lemma 3.5 of [2] it follows that $ma=0$; i.e. $a\in m^\gamma$.

Now suppose J is a right ideal of R and $J\supset m^\gamma$. $mJ\neq(0)$, so there exists $j\in J$ with $mj\neq 0$. It follows directly from the proof of Theorem 3.7 of [2] that there exists $b\in R$ such that $(mjb)^\gamma=m^\gamma$ (in the terminology of [2] mjb and m are D-linearly dependent and $mjb\neq 0$). Then $jb\in J\cap N(m^\gamma)$ and $jb\notin m^\gamma$. Hence $J\cap N(m^\gamma)\supset m^\gamma$.

Again assume $J\supset m^\gamma$, and let $r\in R$ such that $[J:r]\supseteq m^\gamma$. We show $[J:r]\supset m^\gamma$. First suppose $(mr)^\gamma=m^\gamma$. Then $mJ\neq(0)$ and $rJ+m^\gamma\supset m^\gamma$. Let $n\in(rJ+m^\gamma)\cap J$, $n\notin m^\gamma$. We can choose $n=rj$, $j\in J$. Then $n\in J$ implies $j\in[J:r]$. Also $rj=n\notin m^\gamma$ implies $j\notin(mr)^\gamma=m^\gamma$. Hence $[J:r]\supset m^\gamma$. Now suppose $(mr)^\gamma\neq m^\gamma$. Again by Lemma 3.5 of [2] there exists $a\in R$ such that $mra=0$ and $ma\neq 0$. Then $a\in[m^\gamma:r]\subseteq[J:r]$ but $a\notin m^\gamma$. Hence $[J:r]\supset m^\gamma$.

The lemma is proved.

THEOREM 2.3. *If $W(R)\neq R$, then $W(R)$ is the intersection of the set of almost maximal right ideals of R.*

PROOF. Let I be an almost maximal right ideal and let K be the kernel of the associated weakly transitive representation of R, i.e. $K=\{r\in R: Rr\subseteq I\}$. If $n\in N(I)$, $n\notin I$, then $[I:n]=I$. It follows that $K\subseteq [I:n]=I$. Therefore $W(R)$ is contained in the intersection of the set of almost maximal right ideals.

Conversely, if r is in the given intersection, then for any almost maximal right ideal I we have $r\in\bigcap\{m^{\gamma}: m\in R/I\}$ by the lemma. This implies that r is in the kernel of each representation associated with an almost maximal right ideal. Hence $r\in W(R)$. The theorem is proved.

THEOREM 2.4. *If R is a ring and ϕ is a homomorphism of R, then $W(R)\phi\subseteq W(R\phi)$.*

PROOF. Suppose ψ is a weakly transitive representation of $R\phi$. Then $\phi\psi$ is a weakly transitive representation of R. Hence $W(R)\phi\subseteq \operatorname{Ker} \psi$.

THEOREM 2.5. *Suppose R is a weakly transitive ring of linear transformations and J is a (two-sided) ideal of R. Then J is also weakly transitive.*

PROOF. Suppose R acts in the (left) vector space V over a division ring D. Let K and M be as given in Definition A. Clearly M is a (K, J)-submodule of V and M is uniform as J-module. Suppose $\{m_i\}_{i=1}^n$ is a finite D-linearly independent subset of M and $\{y_i\}_{i=1}^n$ is a sequence from M. Let $a\in J$, $a\neq 0$, and let $m\in M$ such that $ma\neq 0$. There exist $\{r_j\}_{j=1}^n$ in R and $\{k_j\}_{j=1}^n$ in k, $k_j\neq 0$, such that $m_ir_j=0$, $j\neq i$ and $m_ir_i=k_im$. Let I be an almost maximal right ideal such that $M=R/I$. Since $m_ir_ia=k_ima\neq 0$, each i, there exist $\{s_i\}_{i=1}^n$ in R, $s_i\notin I$ such that $m_ir_i\,as_i=k_i'm_i$, where $k_i'\in K$, $k_i'\neq 0$, $1\leqq i\leqq n$. Now $\{k_i'm_i\}_{i=1}^n$ is a D-linearly independent subset of M. Hence there exists $r'\in R$, $k\in K$, $k\neq 0$, such that $k_i'm_ir'=ky_i, 1\leqq i\leqq n$. Let $r=\sum_{i=1}^n r_ias_ir'$. Then $r\in J$ and $m_ir=ky_i$, $1\leqq i\leqq n$. Therefore J is weakly transitive.

THEOREM 2.6. *If R is a ring and S is a two-sided ideal of R, then $W(S)=W(R)\cap S$.*

PROOF. Let $S\neq (0)$. Suppose $a\in S$, $a\notin W(R)$. Let ϕ be a weakly transitive representation of R such that $\phi(a)\neq 0$. $\phi(R)$ is a weakly transitive ring and $\phi(S)$ is a nonzero two-sided ideal of $\phi(R)$. By Theorem 2.5, $\phi(S)$ is a weakly transitive ring. Hence $a\notin W(S)$. Therefore $W(S)\subseteq W(R)\cap S$.

Conversely, suppose $a\in S$, $a\notin W(S)$. Let I be an almost maximal right ideal of S such that $Sa\nsubseteq I$. Let $M=S/I$ and let $m\in M$

such that $ma \neq 0$. There exists $s_0 \in S$ such that $m^\gamma = \{s \in S : ms = 0\} = \{s \in S : ms_0 s = 0\} = (ms_0)^\gamma$. Then $ms_0 a \neq 0$. Let $I^* = \{r \in R : m(s_0 r) = 0\}$. I^* is clearly closed under addition. Suppose $r \in I^*$, $t \in R$. If $rt \notin I^*$ then $m(s_0 rt) \neq 0$. Hence there exists $s \in S$ such that $0 \neq m(s_0 rt)s = m(s_0 r)(ts)$, which is absurd. Thus I^* is a right ideal of R. We show that I^* is almost maximal in R.

Let J^* be a right ideal of R such that $J^* \supset I^*$. Let $j \in J^*$, $j \notin I^*$. Then $m(s_0 j) \neq 0$. Hence there exists $s \in S$ such that $m(s_0 js) \neq 0$. Then $js \in S \cap J^*$, $js \notin I^*$. Thus $S \cap J^* \supset S \cap I^* = m^\gamma$. Since m^γ is irreducible in S, it follows immediately that I^* is irreducible in R.

Let $n \in N(m^\gamma)$ and $r \in I^*$. Suppose $m(s_0 nr) \neq 0$. Then there exists $s \in S$ such that $m(s_0 nrs) \neq 0$. Then $m\ (nrs) \neq 0$. Since $n \in N(m^\gamma)$, $rs \notin m^\gamma$. But $m(s_0 rs) = m(s_0 r)s = 0$, which implies $rs \in (ms_0)^\gamma = m^\gamma$, a contradiction. Therefore $N(m^\gamma) \subseteq N(I^*)$. Also $(S \cap J^*) \cap N(m^\gamma) \supset m^\gamma$. If $t \in S \cap J^* \cap N(m^\gamma)$, $t \notin m^\gamma$, then $ms_0 t \neq 0$. Hence $t \notin I^*$, but $t \in J^* \cap N(I^*)$. Thus $J^* \cap N(I^*) \supset I^*$.

Now let $J^* \supset I^*$ and let $r \in R$ such that $rJ^* \subseteq I^*$. If $r \notin I^*$, then $m(s_0 r) \neq 0$. Since $J^* \cap S \supset m^\gamma$, we cannot have $[m(s_0 r)]^\gamma \supseteq J^* \cap S$. Hence there exists $j \in J^* \cap S$ such that $m(s_0 rj) \neq 0$. Then $rj \notin I^*$, a contradiction. Hence $r \in I^*$.

Again suppose $J^* \supset I^*$ and $r \in R$ such that $[J^* : r] \supseteq I^*$. We show $[J^* : r] \supset I^*$. Suppose $[J^* : r] = I^*$. Let $J = J^* \cap S \supset m^\gamma$. Assume first that $[m(s_0 r)]^\gamma = m^\gamma$. Then $m(s_0 rJ) \neq (0)$. Hence $rJ \nsubseteq (ms_0)^\gamma = m^\gamma$, and $rJ + m^\gamma \supset m^\gamma$. Then there exists $n \in (rJ + m^\gamma) \cap J$, $n \notin m^\gamma$. We can choose $n = rj$, $j \in J$. If $s \in m^\gamma$ then $s \in [m(s_0 r)]^\gamma$, and $m(s_0 rs) = 0$. Then $rs \in (ms_0)^\gamma = m^\gamma$. Hence $r(m^\gamma) \subseteq m^\gamma$. Since $rj \notin m^\gamma$, $j \notin m^\gamma$. But $j \in [J^* : r]$. Hence $[J^* : r] \supset I^*$. Now assume that $[m(s_0 r)]^\gamma \neq m^\gamma$, then $[m(s_0 r)]^\gamma \nsubseteq m^\gamma$, so there exists $a \in S$ such that $ms_0 ra = 0$ but $ma \neq 0$. Then $ra \in I^* \subseteq J^*$ implies $a \in [J^* : r]$. But $a \notin m^\gamma = (ms_0)^\gamma$ so $a \notin I^*$. Thus $[J^* : r] \supset I^*$.

It follows that I^* is almost maximal in R. Since $a \notin I^*$, it follows from Theorem 2.3 that $a \notin W(R)$. Therefore $W(R) \cap S \subseteq W(S)$.

THEOREM 2.7. *If R is a ring and n is a positive integer, then $W(R_n) = W(R)_n$.*

PROOF. $W(R_n) \subseteq W(R)_n$. Let $\alpha = (a_{ij}) \in R_n$, $\alpha \notin W(R)_n$. Assume $a_{kl} \notin W(R)$. By Theorem 2.3 there is an almost maximal right ideal I of R such that $a_{kl} \notin I$. Let $I^* = \{(b_{ij}) \in R_n : b_{kj} \in I,\ 1 \leq j \leq n\}$. We show that I^* is an almost maximal right ideal of R_n. Since $\alpha \notin I^*$ it will follow by 2.3 that $\alpha \notin W(R_n)$. For simplicity of notation we give the proof for $k = 1$.

Assume J^* is a right ideal of R_n and $J^* \supset I^*$. Let $\beta = (b_{ij}) \in J^*$,

$\beta \notin I^*$. Suppose $b_{1s} \notin I$. Then there exists $b \in R$ such that $b_{1s}b \in N(I)$, $b_{1s}b \notin I$. Let $\gamma = (c_{ij})$, where $c_{s1} = b$ and $c_{ij} = 0$, $i \neq s$ or $j \neq 1$. Then $\beta\gamma = \rho = (r_{ij})$, where $r_{11} = b_{1s}b$ and $r_{1j} = 0$, $j > 1$, $\rho \in N(I^*)$, $\rho \notin I^*$. Hence $J^* \cap N(I^*) \supset I^*$. It is now clear that if J_1^* and J_2^* are right ideals of R_n and $J_1^* \supset I^*$, $J_2^* \supset I^*$, then $J_1^* \cap J_2^* \supset I^*$.

Again assume that $J^* \supset I^*$ and let $\beta = (b_{ij}) \in R_n$, $\beta \notin I^*$. We show $\beta J^* \not\subseteq I^*$. Assume $b_{1s} \notin I$. Clearly, if $s > 1$. then $\beta J^* \not\subseteq I^*$. Suppose $s = 1$. Let $J = \{r \in R$: there exists $(c_{ij}) \in J^*$ with $c_{11} = r\}$. Then $J \supset I$. So there exists $r \in J$ such that $b_{11}r \notin I$. Choose $\gamma = (c_{ij}) \in J^*$ such that $c_{11} = r$, $c_{ij} = 0$ for $i > 1$. Then $\beta\gamma \notin I^*$.

Let $J^* \supset I^*$ and suppose $\beta = (b_{ij}) \in R_n$ such that $\beta I^* \subseteq J^*$. If $b_{11} \in I$, let $\gamma = (c_{ij}) \in R_n$, $\gamma \notin I^*$, such that $c_{ij} = 0$ for $i > 1$. Then $\beta\gamma \in I^* \subset J^*$ implies $[J^* : \beta] \supset I^*$. Suppose $b_{11} \notin I$. Let $J = \{r \in R$: there exists $(c_{ij}) \in J^*$ with $c_{11} = r$, $c_{ij} = 0$ for $j > 1\}$. J is a right ideal of R and $J \supset I$, where the inclusion is proper because, as shown above, there is an element $(r_{ij}) \in J^* \cap N(I^*)$, $(r_{ij}) \notin I^*$, such that $r_{ij} = 0$ for $j > 1$. Also $b_{11}I \subseteq J$. Hence $[J : b_{11}] \supset I$, because I is almost maximal. Let $c \in [J : b_{11}]$, $c \notin I$. Choose $\gamma = (c_{ij}) \in R_n$ such that $c_{11} = c$, $c_{ij} = 0$ for $i \neq 1$ or $j \neq 1$. Then $\beta\gamma \in J^*$ but $\gamma \notin I^*$. Hence again $[J^* : \beta] \supset I^*$. It now follows that I^* is an almost maximal right ideal of R_n

$W(R)_n \subseteq W(R_n)$. It is sufficient to show that any matrix in $W(R)_n$ with only one nonzero row must be in $W(R_n)$. For ease of notation we give the proof in the case where the nonzero row is the first row. Hence suppose $\alpha = (a_{ij}) \in R_n$, $\alpha \notin W(R_n)$, and $a_{ij} = 0$ for $i > 1$. We show $\alpha \notin W(R)_n$.

There is an almost maximal right ideal J^* of R_n such that $\alpha \notin J^*$. There exists $\beta = (b_{ij}) \in N(J^*) \cap \alpha R_n$, $\beta \notin J^*$. If we show $\beta \notin W(R)_n$ it will follow that $\alpha \notin W(R)_n$. Let $\gamma = (c_{ij})$ be such that $c_{11} = b_{11}$, $c_{ij} = 0$ for $i \neq 1$ or $j \neq 1$. Then $\gamma\beta = \beta^2$. $\beta^2 \notin J^*$ because $N(J^*)/J^*$ is an integral domain. Hence $\gamma \notin J^*$.

Let $I^* = \{\rho \in R_n : \gamma\rho \in J^*\}$. By Lemma 2.2, I^* is an almost maximal right ideal of R_n. If $\rho \in I^*$, then $\gamma\rho \in J^*$. Since $\beta \in N(J^*)$, $\beta\gamma\rho \in J^*$. But $\beta\gamma\rho = \gamma^2\rho$, hence $\gamma\rho \in I^*$. Thus $\gamma \in N(I^*)$, $\gamma \notin I^*$.

Let $I = \{r \in R$: there exists $(r_{ij}) \in I^*$ with $r_{11} = r\}$. Then I is a right ideal of R. Now $b_{11} \notin I$; for suppose $\rho = (r_{ij}) \in I^*$ with $r_{11} = b_{11}$. We can assume $r_{ij} = 0$ for $i > 1$. Then $\gamma^2 = \rho\gamma \in I^*$, which is impossible because $\gamma \in N(I^*)$, $\gamma \notin I^*$. We now show that I is an almost maximal right ideal of R, from which it will follow that $b_{11} \notin W(R)$, and the theorem will be proved.

Suppose J is a right ideal of R and $J \supset I$. Let $r \in J$, $r \notin I$. Define $\rho = (r_{ij})$ with $r_{11} = r$, $r_{ij} = 0$ for $i \neq 1$ or $j \neq 1$. Then $\rho \notin I^*$. Hence there exists $\tau = (t_{ij}) \in R_n$ such that $\rho\tau \in N(I^*)$ but $\rho\tau \notin I^*$. Then $rt_{11} \in N(I)$.

But $rt_{11} \notin I$. For suppose $\mu = (m_{ij}) \in I^*$ with $m_{11} = rt_{11}$. Again we can suppose $m_{ij} = 0$ for $i > 1$. Then $\rho\tau\gamma = \mu\gamma$, which is a contradiction because $\rho\tau\gamma \notin I^*$ while $\mu\gamma \in I^*$. Thus $rt_{11} \in J \cap N(I)$, $rt_{11} \notin I$, and $J \cap N(I) \supset I$.

We show now that $I^* = \{(r_{ij}): r_{1j} \in I, 1 \leqq j \leqq n\}$. Suppose $\rho = (r_{ij}) \in I^*$. Then clearly $r_{1j}r \in I$ for each $r \in R$, $1 \leqq j \leqq n$. But this implies, by the above argument, that $r_{1j} \in I$. Conversely, suppose $r_{1j} \in I$, $1 \leqq j \leqq n$. If $\rho = (r_{ij}) \notin I^*$, then there exists $\tau \in R_n$ such that $\rho\tau \in N(I^*)$, $\rho\tau \notin I^*$. Let $\rho\tau = \sigma = (s_{ij})$. Again we may assume $s_{ij} = 0$ for $i > 1$. By the argument above, $s_{11} \notin I$, a contradiction.

Assume J_1 and J_2 are right ideals of R and $J_1 \supset I$, $J_2 \supset I$. Let $J_k^* = \{(r_{ij}): r_{1j} \in J_k, 1 \leqq j \leqq n\}$, $k = 1, 2$. Then J_1^* and J_2^* are right ideals of R_n and $J_k^* \supset I^*$, $k = 1, 2$. Hence $J_1^* \cap J_2^* \supset I^*$, from which it follows easily that $J_1 \cap J_2 \supset I$.

Also, if $J \supset I$ and $r \in R$ such that $rI \subseteq J$, let $\sigma = (s_{ij})$ with $s_{11} = r$, and $s_{ij} = 0$ for $i \neq 1, j \neq 1$. Let $J^* = \{(r_{ij}): r_{1j} \in J, 1 \leqq j \leqq n\}$. Then $\sigma I^* \subseteq J^*$. Hence $[J^*:\sigma] \supset I^*$, which clearly implies $[J:r] \supset I$. The theorem is proved.

References

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North Carolina State University at Raleigh and
The University of North Carolina at Chapel Hill