1. Introduction. In [2] a weakly transitive ring of linear transformations is defined as follows:

Definition A. Let $V$ be a left vector space over a division ring $D$ and let $R$ be a ring of linear transformations of $V$. Consider $V$ as right $R$-module. Then $R$ is weakly transitive provided there is a right order $K$ in $D$ and a $(K, R)$-submodule $M$ of $V$ such that $M$ is uniform as $R$-module, $DM = V$, and such that if $\{m_i\}_{i=1}^n$ is a finite $D$-linearly independent subset of $M$ and if $\{y_i\}_{i=1}^n$ is a sequence from $M$, then there exists $r \in R$, $k \in K$, $k \neq 0$, such that $m_i r = k y_i$, $1 \leq i \leq n$.

By a weakly transitive representation of a ring $R$ we mean a homomorphism of $R$ onto a weakly transitive ring.

Definition 1.1. If $R$ is a ring the weak radical, $W(R)$, of $R$ is the intersection of the set of all ideals which are kernels of weakly transitive representations. If $R$ has no weakly transitive representation, then $W(R) = R$.

Of importance in studying weakly transitive rings is the notion of an almost maximal right ideal, which is also defined in [2].

Definition B. If $I$ is a proper right ideal of a ring $R$ then $I$ is almost maximal provided that

1. if $J_1$ and $J_2$ are right ideals of $R$ and $J_1 \cap J_2 = I$, then $J_1 = I$ or $J_2 = I$, i.e. $I$ is irreducible,
2. if $a \in R$ and $[I:a] \supset I$, then $a \in I$,
3. if $J$ is a right ideal of $R$, $J \supset I$, then $N(I) \cap J \supset I$, where $N(I) = \{r \in R : r I \subseteq I\}$, and if $a \in R$ such that $[J:a] \supset I$ then $[J:a] \supset I$.

It was shown in [2] that if $I$ is an almost maximal right ideal of $R$, then $M = R/I$ is a uniform right $R$-module with centralizer $K = N(I)/I$. $M$ has extended centralizer $D$ [see 2, p. 67] in which $K$ is a right order, and $M$ can be extended to a $(D, R)$-module $V$ such that $DM = V$ and such that $R$ induces a weakly transitive ring of linear transformations of the $D$-space $V$.

In this paper we observe that $W(R/W(R)) = (0)$ and prove that $W(R)$ is the intersection of the set of almost maximal right ideals of $R$. From the fact that a (two-sided) ideal of a weakly transitive ring is also weakly transitive we deduce that if $S$ is an ideal of $R$ then $W(S) = W(R) \cap S$. Finally, if $n$ is a positive integer then $W(R_n) = W(R)^n$. 

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1 By $[I : a]$, we mean the set $\{r \in R : ar \in I\}$. 

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It is interesting to note that in several instances the usual proofs of the analogues of our theorems in the Jacobson structure theory (see, for example, [1, p. 10]) make very strong use of the notion of quasi-regularity. Since in the weak radical theory we have no analogue of this notion the proofs which we give suggest proofs of the classical theorems which do not depend on quasi-regularity. The authors have been unable to devise a proof of the equality of the left and right radicals without using quasi-regularity and this problem for the weak radical remains open.

From the definition of $W(R)$ it follows that if $W(R) = (0)$, then $R$ is a subdirect sum of weakly transitive rings. It is likely that this hypothesis of "weak semisimplicity" will serve in many cases where a ring is assumed to be semisimple.

2. The proof of the following theorem is straightforward and will be omitted.

**Theorem 2.1.** If $R$ is a ring then $W(R/W(R)) = (0)$.

**Lemma 2.2.** If $I$ is an almost maximal right ideal of a ring $R$ and $M = R/I$, then for each $m \in M$, $m \neq 0$, the right ideal $m^r = \{r \in R : mr = 0\}$ is almost maximal.

**Proof.** $mR$ is a submodule of $M$ and hence is uniform. Since $mR \simeq R/m^r$, $m^r$ is an irreducible right ideal. Suppose $a \in R$ and $(m^r : a) = (ma)^r \supseteq m^r$. By Lemma 3.5 of [2] it follows that $ma = 0$; i.e. $a \in m^r$.

Now suppose $J$ is a right ideal of $R$ and $J \supseteq m^r$. $mJ \neq (0)$, so there exists $j \in J$ with $mj \neq 0$. It follows directly from the proof of Theorem 3.7 of [2] that there exists $b \in R$ such that $(mjb)^r = m^r$ (in the terminology of [2] $mjb$ and $m$ are $D$-linearly dependent and $mjb \neq 0$). Then $jb \in J \cap N(m^r)$ and $jb \notin m^r$. Hence $J \cap N(m^r) \supsetneq m^r$.

Again assume $J \supseteq m^r$, and let $r \in R$ such that $[J : r] \supseteq m^r$. We show $[J : r] \supseteq m^r$. First suppose $(mr)^r = m^r$. Then $mJ \neq (0)$ and $rJ + m^r \supseteq m^r$. Let $n \in (rJ + m^r) \cap J$, $n \notin m^r$. We can choose $n = rj$, $j \in J$. Then $n \in J$ implies $j \in [J : r]$. Also $rj = n \notin m^r$ implies $j \notin (mr)^r = m^r$. Hence $[J : r] \supsetneq m^r$. Now suppose $(mr)^r \neq m^r$. Again by Lemma 3.5 of [2] there exists $a \in R$ such that $mra = 0$ and $ma \neq 0$. Then $a \in [m^r : r] \subseteq [J : r]$ but $a \notin m^r$. Hence $[J : r] \supsetneq m^r$.

The lemma is proved.

**Theorem 2.3.** If $W(R) \neq R$, then $W(R)$ is the intersection of the set of almost maximal right ideals of $R$.

**Proof.** Let $I$ be an almost maximal right ideal and let $K$ be the
kernel of the associated weakly transitive representation of $R$, i.e. $K = \{ r \in R : Rr \subseteq I \}$. If $n \in N(I)$, $n \in I$, then $[I:n] = I$. It follows that $K \subseteq [I:n] = I$. Therefore $W(R)$ is contained in the intersection of the set of almost maximal right ideals.

Conversely, if $r$ is in the given intersection, then for any almost maximal right ideal $I$ we have $r \in \bigcap \{ m : m \in R/I \}$ by the lemma. This implies that $r$ is in the kernel of each representation associated with an almost maximal right ideal. Hence $r \in W(R)$. The theorem is proved.

**Theorem 2.4.** If $R$ is a ring and $\phi$ is a homomorphism of $R$, then $W(R) \phi \subseteq W(R \phi)$.

**Proof.** Suppose $\psi$ is a weakly transitive representation of $R \phi$. Then $\phi \psi$ is a weakly transitive representation of $R$. Hence $W(R) \phi \subseteq \text{Ker } \psi$.

**Theorem 2.5.** Suppose $R$ is a weakly transitive ring of linear transformations and $J$ is a (two-sided) ideal of $R$. Then $J$ is also weakly transitive.

**Proof.** Suppose $R$ acts in the (left) vector space $V$ over a division ring $D$. Let $K$ and $M$ be as given in Definition A. Clearly $M$ is a $(K, J)$-submodule of $V$ and $M$ is uniform as $J$-module. Suppose $\{ m_i \}_{i=1}^n$ is a finite $D$-linearly independent subset of $M$ and $\{ y_i \}_{i=1}^n$ is a sequence from $M$. Let $a \in J$, $a \neq 0$, and let $m \in M$ such that $ma \neq 0$. There exist $\{ r_i \}_{i=1}^n$ in $R$ and $\{ k_j \}_{j=1}^n$ in $K$, $k_j \neq 0$, such that $m_i r_j = 0$, $j \neq i$ and $m_i r_i = k_j m_i$. Let $I$ be an almost maximal right ideal such that $M = R/I$. Since $m_i r_i a = k_i ma \neq 0$, each $i$, there exist $\{ s_i \}_{i=1}^n$ in $R$, $s_i \in I$ such that $m_i r_i s_i = k'_i m_i$, where $k'_i \in K$, $k'_i \neq 0$, $1 \leq i \leq n$. Now $\{ k'_i m_i \}_{i=1}^n$ is a $D$-linearly independent subset of $M$. Hence there exists $r' \in R$, $r' \in K$, $r' \neq 0$, such that $k'_i m_i r'_i = k y_i$, $1 \leq i \leq n$. Let $r = \sum_{i=1}^n r_i a s_i r'_i$. Then $r \in J$ and $m_i r_i = k y_i$, $1 \leq i \leq n$. Therefore $J$ is weakly transitive.

**Theorem 2.6.** If $R$ is a ring and $S$ is a two-sided ideal of $R$, then $W(S) = W(R) \cap S$.

**Proof.** Let $S \neq (0)$. Suppose $a \in S$, $a \in W(R)$. Let $\phi$ be a weakly transitive representation of $R$ such that $\phi(a) \neq 0$. $\phi(R)$ is a weakly transitive ring and $\phi(S)$ is a nonzero two-sided ideal of $\phi(R)$. By Theorem 2.5, $\phi(S)$ is a weakly transitive ring. Hence $a \in W(S)$. Therefore $W(S) \subseteq W(R) \cap S$.

Conversely, suppose $a \in S$, $a \in W(S)$. Let $I$ be an almost maximal right ideal of $S$ such that $Sa \subseteq I$. Let $M = S/I$ and let $m \in M$
such that $ma \neq 0$. There exists $s_0 \in S$ such that $m^\gamma = \{s \in S : ms = 0\} = \{s \in S : ms_0s = 0\} = (ms_0)^\gamma$. Then $ms_0a \neq 0$. Let $I^* = \{r \in R : m(s_0r) = 0\}$. $I^*$ is clearly closed under addition. Suppose $rt \in I^*$, $t \in R$. If $rt \in I^*$ then $m(s_0rt) \neq 0$. Hence there exists $s \in S$ such that $0 \neq m(s_0rt)s = m(s_0r)(ts)$, which is absurd. Thus $I^*$ is a right ideal of $R$. We show that $I^*$ is almost maximal in $R$.

Let $J^*$ be a right ideal of $R$ such that $J^* \supset I^*$. Let $j \in J^*$, $j \in I^*$. Then $m(s_0j) \neq 0$. Hence there exists $s \in S$ such that $m(s_0js) \neq 0$. Then $js \in S \cap J^*$, $js \in I^*$. Thus $S \cap J^* \supset S \cap I^* = m^\gamma$. Since $m^\gamma$ is irreducible in $S$, it follows immediately that $I^*$ is irreducible in $R$.

Let $n \in N(m^\gamma)$ and $r \in I^*$. Suppose $m(s_0nr) \neq 0$. Then there exists $s \in S$ such that $m(s_0rs) \neq 0$. Then $m((rs)n) \neq 0$. Since $n \in N(m^\gamma)$, $rs \in m^\gamma$. But $m(s_0rs) = m(s_0r)s = 0$, which implies $rs \in (ms_0)^\gamma = m^\gamma$, a contradiction. Therefore $N(m^\gamma) \subseteq N(I^*)$. Also $(S \cap J^*) \cap N(m^\gamma) \subseteq m^\gamma$.

If $t \in S \cap J^* \cap N(m^\gamma)$, $t \in m^\gamma$, then $ms_0t \neq 0$. Hence $t \in I^*$, but $t \in J^* \cap N(I^*)$. Thus $J^* \cap N(I^*) \subseteq I^*$.

Now let $J^* \supset I^*$ and let $r \in R$ such that $rJ^* \subseteq I^*$. If $r \in I^*$, then $m(s_0r) \neq 0$. Since $J^* \cap S \supset m^\gamma$, we cannot have $[m(s_0r)]^\gamma \supset J^* \cap S$. Hence there exists $j \in J^* \cap S$ such that $m(s_0rj) \neq 0$. Then $rj \in I^*$, a contradiction. Hence $r \notin I^*$.

Again suppose $J^* \supset I^*$ and $r \in R$ such that $[J^* : r] \supset I^*$. We show $[J^* : r] \supset I^*$. Suppose $[J^* : r] = I^*$. Let $J = J^* \cap S \supset m^\gamma$. Assume first that $[m(s_0r)]^\gamma = m^\gamma$. Then $m(s_0rj) \neq (0)$. Hence $rJ \subseteq (ms_0)^\gamma = m^\gamma$, and $rJ + m^\gamma \supset m^\gamma$. Then there exists $n \in (rJ + m^\gamma) \cap J$, $n \notin m^\gamma$. We can choose $n = rj$, $j \in J$. If $s \in m^\gamma$ then $s \in [m(s_0r)]^\gamma$, and $m(s_0rs) = 0$. Then $rs \in (ms_0)^\gamma = m^\gamma$. Hence $r(m^\gamma) \subseteq m^\gamma$. Since $rj \in m^\gamma$, $j \in m^\gamma$. But $j \in [J^* : r]$. Hence $[J^* : r] \supset I^*$. Now assume that $[m(s_0r)]^\gamma \neq m^\gamma$, then $[m(s_0r)]^\gamma \subseteq m^\gamma$, so there exists $a \in S$ such that $ms_0a = 0$ but $ma \neq 0$. Then $ra \in I^* \subseteq J^*$ implies $a \in [J^* : r]$. But $a \in m^\gamma = (ms_0)^\gamma$ so $a \notin I^*$. Thus $[J^* : r] \supset I^*$.

It follows that $I^*$ is almost maximal in $R$. Since $a \in I^*$, it follows from Theorem 2.3 that $a \in W(R)$. Therefore $W(R) \cap S \subseteq W(S)$.

**Theorem 2.7.** If $R$ is a ring and $n$ is a positive integer, then $W(R_n) = W(R)_n$.

**Proof.** $W(R_n) \subseteq W(R)_n$. Let $\alpha = (a_{ij}) \in R_n$, $\alpha \in W(R)_n$. Assume $a_{kl} \in W(R)$. By Theorem 2.3 there is an almost maximal right ideal $I$ of $R$ such that $a_{kl} \in I$. Let $I^* = \{(b_{ij}) \in R_n : b_{ij} \in I, 1 \leq j \leq n\}$. We show that $I^*$ is an almost maximal right ideal of $R_n$. Since $\alpha \in I^*$ it will follow by 2.3 that $\alpha \in W(R_n)$. For simplicity of notation we give the proof for $k = 1$.

Assume $J^*$ is a right ideal of $R_n$ and $J^* \supset I^*$. Let $\beta = (b_{ij}) \in J^*$,
Suppose $b_{1s} \in I$. Then there exists $b \in R$ such that $b_{1s}b \in N(I)$, $b_{1s}b \in I$. Let $\gamma = (c_{ij})$, where $c_{ii} = b$ and $c_{ij} = 0$, $i \neq s$ or $j \neq 1$. Then $\beta \gamma = \rho = (r_{ij})$, where $r_{11} = b_{1s}b$ and $r_{ij} = 0$, $j > 1$, $\rho \in N(I^*)$, $\rho \in I^*$. Hence $J^* \cap N(I^*) \subseteq I^*$. It is now clear that if $J_1^*$ and $J_2^*$ are right ideals of $R_n$ and $J_1^* \supseteq I^*$, $J_2^* \supseteq I^*$, then $J_1^* \cap J_2^* \supseteq I^*$.

Again assume that $J^* \supseteq I^*$ and let $\beta = (b_{ij}) \in R_n$, $\beta \in I^*$. We show $\beta J^* \subseteq I^*$. Assume $b_{1s} \in I$. Clearly, if $s > 1$, then $\beta J^* \subseteq I^*$. Suppose $s = 1$. Let $J = \{r \in R: \text{there exists } (c_{ij}) \in J^* \text{ with } c_{11} = r\}$. Then $J \supseteq I$. So there exists $r \in J$ such that $b_{11} \in I$. Choose $\gamma = (c_{ij}) \in J^*$ such that $c_{11} = r$, $c_{ij} = 0$ for $i > 1$. Then $\beta \gamma \in I^*$.

Let $J^* \supseteq I^*$ and suppose $\beta = (b_{ij}) \in R_n$ such that $\beta I^* \subseteq J^*$. If $b_{11} \in I$, let $\gamma = (c_{ij}) \in R_n$, $\gamma \in I^*$, such that $c_{ij} = 0$ for $i > 1$. Then $\beta \gamma \in I^* \subseteq J^*$. Suppose $b_{11} \in I$. Let $J = \{r \in R: \text{there exists } (c_{ij}) \in J^* \text{ with } c_{11} = r, c_{ij} = 0 \text{ for } j > 1\}$. $J$ is a right ideal of $R$ and $J \supseteq I$, where the inclusion is proper because, as shown above, there is an element $(r_{ij}) \in J^* \cap N(I^*)$, $(r_{ij}) \in I^*$, such that $r_{ij} = 0$ for $j > 1$. Also $b_{11}I \subseteq J$. Hence $[J:b_{11}] \supseteq I$, because $I$ is almost maximal. Let $c \in [J:b_{11}]$, $c \in I$. Choose $\gamma = (c_{ij}) \in R_n$ such that $c_{11} = c$, $c_{ij} = 0$ for $i \neq 1$ or $j \neq 1$. Then $\beta \gamma \in J^*$ but $\gamma \in I^*$. Hence again $[J^*:\beta] \supseteq I^*$. It now follows that $I^*$ is an almost maximal right ideal of $R_n$.

$W(R)_n \subseteq W(R_n)$. It is sufficient to show that any matrix in $W(R)_n$ with only one nonzero row must be in $W(R_n)$. For ease of notation we give the proof in the case where the nonzero row is the first row. Hence suppose $\alpha = (a_{ij}) \in R_n$, $\alpha \in W(R_n)$, and $a_{ij} = 0$ for $i > 1$. We show $\alpha \in W(R_n)$.

There is an almost maximal right ideal $J^*$ of $R_n$ such that $\alpha \in J^*$. There exists $\beta = (b_{ij}) \in N(J^*) \cap \alpha R_n$, $\beta \in J^*$. If we show $\beta \in W(R_n)$ it will follow that $\alpha \in W(R_n)$. Let $\gamma = (c_{ij})$ be such that $c_{11} = b_{11}$, $c_{ij} = 0$ for $i \neq 1$ or $j \neq 1$. Then $\gamma \beta = \beta^2$. $\beta^2 \in J^*$ because $N(J^*)/J^*$ is an integral domain. Hence $\gamma \in J^*$.

Let $I^* = \{\rho \in R_n: \gamma \rho \in J^*\}$. By Lemma 2.2, $I^*$ is an almost maximal right ideal of $R_n$. If $\rho \in I^*$, then $\gamma \rho \in J^*$. Since $\beta \in N(J^*)$, $\beta \gamma \rho \in J^*$. But $\beta \gamma \rho = \gamma^2 \rho$, hence $\gamma \rho \in I^*$. Thus $\gamma \in N(I^*)$, $\gamma \in I^*$.

Let $I = \{r \in R: \text{there exists } (r_{ij}) \in I^* \text{ with } r_{11} = r\}$. Then $I$ is a right ideal of $R$. Now $b_{11} \in I$; for suppose $\rho = (r_{ij}) \in I^*$ with $r_{11} = b_{11}$. We can assume $r_{ij} = 0$ for $i > 1$. Then $\gamma^2 = \rho \gamma \in I^*$, which is impossible because $\gamma \in N(I^*)$, $\gamma \in I^*$. We now show that $I$ is an almost maximal right ideal of $R$, from which it will follow that $b_{11} \in W(R)$, and the theorem will be proved.

Suppose $J$ is a right ideal of $R$ and $J \supseteq I$. Let $r \in J$, $r \in I$. Define $\rho = (r_{ij})$ with $r_{11} = r$, $r_{ij} = 0$ for $i \neq 1$ or $j \neq 1$. Then $\rho \in I^*$. Hence there exists $\tau = (t_{ij}) \in R_n$ such that $\rho \tau \in N(I^*)$ but $\rho \tau \notin I^*$. Then $rt_{11} \in N(I)$. 

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But $rt_{11} \in I$. For suppose $\mu = (m_{ij}) \in I^*$ with $m_{11} = rt_{11}$. Again we can suppose $m_{ij} = 0$ for $i > 1$. Then $\rho r = \mu \gamma$, which is a contradiction because $\rho r \gamma \in I^*$ while $\mu \gamma \in I^*$. Thus $rt_{11} \in J \cap N(I)$, $rt_{11} \in I$, and $J \cap N(I) \supset I$.

We show now that $I^* = \{(r_{ij}) : r_{ij} \in I, 1 \leq j \leq n\}$. Suppose $\rho = (r_{ij}) \in I^*$. Then clearly $r_{ij} \in I$ for each $r \in R$, $1 \leq j \leq n$. But this implies, by the above argument, that $r_{ij} \in I$. Conversely, suppose $r_{ij} \in I$, $1 \leq j \leq n$. If $\rho = (r_{ij}) \in I^*$, then there exists $\tau \in R_n$ such that $\rho \tau \in N(I^*)$, $\rho \tau \in I^*$. Let $\rho \tau = \sigma = (s_{ij})$. Again we may assume $s_{ij} = 0$ for $i > 1$. By the argument above, $s_{11} \in I$, a contradiction.

Assume $J_1$ and $J_2$ are right ideals of $R$ and $J_1 \supset I$, $J_2 \supset I$. Let $J_k^* = \{(r_{ij}) : r_{ij} \in J_k, 1 \leq j \leq n\}$, $k = 1, 2$. Then $J_1^*$ and $J_2^*$ are right ideals of $R_n$ and $J_k^* \supset I^*$, $k = 1, 2$. Hence $J_1^* \cap J_2^* \supset I^*$, from which it follows easily that $J_1 \cap J_2 \supset I$.

Also, if $J \supset I$ and $r \in R$ such that $rI \subseteq J$, let $\sigma = (s_{ij})$ with $s_{11} = r$, and $s_{ij} = 0$ for $i \neq 1, j \neq 1$. Let $J^* = \{(r_{ij}) : r_{ij} \in J, 1 \leq j \leq n\}$. Then $\sigma I^* \subseteq J^*$. Hence $[J^* : \sigma] \supset I^*$, which clearly implies $[J : r] \supset I$. The theorem is proved.

References


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