REPRESENTATIONS OF POLYCYCLIC GROUPS

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L. Auslander [1] has recently shown that every polycyclic group[1] has a faithful representation in $GL(n, Z)$ for some n, thus solving a problem of P. Hall [2]. His proof involves considerable knowledge of the theory of Lie groups. Since the result obtained is purely algebraic, it is of interest to find a purely algebraic proof of it. It struck me that the proof of Ado's theorem [5] could be adapted to this purpose and I will show here that this is indeed the case. I would like to thank J. Thompson and J. Alperin for calling this problem to my attention.

Recall that a matrix (a_{ij}) is called uni-triangular if $a_{ij}=0$ for $j<i$ and $a_{ii}=i$ for all i. These form a nilpotent subgroup $T_n(Z)$ of $\mathrm{GL}(n, Z)$.

THEOREM. *Let G be a group and N a normal subgroup of G such that N is finitely generated, torsion free and nilpotent and G/N is finitely generated free abelian. Then G has a faithful representation $\rho: G \to \mathrm{GL}(r, Z)$ such that $\rho(N) \subset T_r(Z)$.*

Since every polycyclic group has a subgroup of finite index of this type [2], [4], we can take induced representations (cf. [1], [2]) and get

COROLLARY. *Every polycylic group has a faithful representation in $\mathrm{GL}(n, Z)$ for some n.*

Proof of the Theorem. The theorem is known for the case $G=N$ [2, Theorem 7.5], [3, Theorem 5.2]. A simple proof is indicated in the remark below. Choose a subgroup H of G with $N \subset H$, $G/H \approx Z$. By induction on the rank of G/N we can assume that H has a representation $\rho: H \to GL(r, Z)$ of the required type. This ρ gives a ring homomorphism $\rho: ZH \to M_r(Z)$. Call its kernel K. Let L be the 2-sided ideal of ZH generated by all $n-1$ with $n \in N$. Since N is normal in H, the identity $(n-1)g=g(g^{-1}ng-1)$ shows that L^{r+1} is generated as a left ideal by all products $(n_0-1) \cdots (n_r-1)$. Therefore $\rho(L^{r+1})=0$ so $L^{r+1} \subset K$. Let $J_i=(L+K)^{r+1} \subset K$. Since H is finitely generated, we see as in [5, Exposé 8, §1] that ZH/J_1 is finitely generated as an abelian group. Let J/J_1 be the torsion subgroup of ZH/J_1. Clearly J

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[1] A group is called polycyclic if it is built up from cyclic groups by a finite number of extensions.

is a 2-sided ideal of ZH and $J\subset K$ since ZH/K is torsion free. Since ρ is faithful, H acts faithfully on ZH/K and thus on ZH/J. Since $L^{r+1}\subset J_1\subset J$, we can choose a base for ZH/J so that N is represented by uni-triangular matrices. (Note that there exist elements fixed by N, e.g., those in the last nonzero subgroup $I_N^i\cdot ZH/J$. Choose e_1 fixed under N and not divisible. Repeat modulo e_1, etc.) Now choose $a\in G$ representing a generator of $G/H=Z$. Let $\alpha(x)=axa^{-1}$ for $x\in H$. Then $x^{-1}\alpha(x)=[x^{-1}, a]\in[G, G]\subset N$ so $\alpha(x)-x=x(x^{-1}\alpha(x)-1)\in L$. Thus $(\alpha-1)ZH\subset L$ so $L+K$ is α stable and therefore so are J_1 and J. We can now extend the action of H on ZH/J to an action of G by letting a act as α. This clearly preserves the relations $aha^{-1}=\alpha(h)$ which define G in terms of H and α. Finally take the direct sum of this representation with a faithful representation of $G/H\approx Z$, e.g., by $Z\approx T_2(Z)$. This gives the required representation of G.

Remark. The case $G=N$ which we took as known above can also be handled directly by the same method. We can again find $H\triangleleft G$, $G/H\approx Z$ and by induction on the rank of G, we can assume that H has a representation of the required type. Proceeding as above we get a representation of G on ZH/J where $I_H^m\subset J$ for some m. The only difficulty is to show that some power of I_G annihilates ZH/J. We show, in fact, that for every m, there is a k with $I_G^k\cdot ZH\subset I_H^m$. For this we need not assume G torsion free. If G is abelian, then $G=H\times Z$ and the result is trivial. We now use induction on the class of nilpotence of G. Let A be the last nontrivial term of the lower central series of G. Then $A\subset[G, G]\subset H$ since G is assumed not abelian. By induction, $I_{G/A}^k\cdot Z[H/A]\subset I_{H/A}^m$ or $I_G^k\cdot ZH\subset I_H^m+ZHI_A$ since ZHI_A is the kernel of $ZG\to Z[G/A]$. Now α is trivial on A so ZH is a left ZG, right ZA-bimodule. If $I_G^l\cdot ZH\subset I_H^m+ZHI_A^r$ then $I_G^{l+k}\cdot ZH\subset I_H^m+I_G^k\cdot(ZHI_A^r)\subset I_H^m+[I_H^m+ZHI_A]I_A^r=I_H^m+ZHI_A^{r+1}$. Eventually, we get to $r=m$ and are finished.

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