A LIMIT-POINT CRITERION FOR NONOSCILLATORY STURM-LIOUVILLE DIFFERENTIAL OPERATORS[1]

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The main point of the present paper is to derive a limit-point criterion from which the criteria of Weyl [6], Friedrichs [3] and Sears [4] follow as special cases. This limit-point criterion is an immediate consequence of a Sturm comparison theorem. It is shown to be equivalent to the criterion of Weyl [6] when use is made of a Liouville-type transformation formula.

Consider the Sturm-Liouville differential operator

$$Ly = -(py')' + qy, \qquad X^- < x < X^+, \tag{1}$$

where, in the open interval (X^-, X^+), $p(x)$ and $q(x)$ are real and continuous and $p(x) > 0$.

Let x_0 be any point where

$$X^- < x_0 < X^+.$$

DEFINITION (WEYL [6]). L is of limit-point type (abbreviated LP) at X^+ if for some constant λ_0 not all solutions of $Ly = \lambda_0 y$ are in $L^2(x_0, X^+)$.

A corresponding definition applies for L being LP at X^-.

We will freely use the fact, proved by Weyl [6, p. 27], [7, p. 415] (see also [2]), that if all solutions of $Ly = \lambda y$ are in L^2 for some $\lambda = \lambda_0$, then this is also the case for every λ in the complex plane.

THEOREM. *L defined by*

$$Ly = -(py')' + qy$$

is LP *at X^+ if there is a comparison operator*

$$\hat{L}v = -(pv')' + \hat{q}v$$

such that (i) *$\hat{L}$ is* LP *and nonoscillatory at X^+ and*
(ii) *$q - \hat{q}$ is bounded below at X^+.*

PROOF. By hypothesis (i) there is a solution, v, of $\hat{L}v = 0$ which is strictly positive and not in L^2 on some interval (x_0, X^+).

By hypothesis (ii) there is a constant λ_0 such that for all $x \geqq x_0$,

$$q - \hat{q} \geqq \lambda_0.$$

Presented to the Society, September 1, 1966; received by the editors June 24, 1966.

[1] This research was completed under U. S. Navy Contract No. Nonr 3360 (01).

Define y to be the solution of $Ly=\lambda_0 y$ which satisfies the initial conditions

$$y(x_0) = v(x_0), \qquad y'(x_0) = v'(x_0).$$

Then by a Sturm comparison theorem (see, for example, [1] or [5])

$$y(x) \geqq v(x) \qquad \text{for } x \geqq x_0.$$

Hence $y(x)$ is not in $L^2(x_0, X^+)$. Q.E.D.

It is easily seen that the above theorem is also valid if one replaces X^+, wherever it occurs, by X^-. To verify this it is sufficient to simply apply the above theorem to the problem obtained after the change of variables $s=-x$.

We now show that the LP criteria of Weyl [6], Friedrichs [3] and Sears [4] follow from the above by proper selection of the comparison operator.

For the first two LP criteria let $\hat{q}=0$. The solutions of $\hat{L}v=0$ are then linear combinations of $v_1=1$ and $v_2=\int_{x0}^{x}(p(s))^{-1}ds$. Clearly $\hat{L}$ is nonoscillatory. Furthermore, $\hat{L}$ will be LP at X^+ if either $X^+=\infty$ (since then $v_1=1$ is not in L^2) or if $X^+<\infty$ and v_2 is not in L^2. Thus we have rederived the following:

Corollary 1 (Weyl [7, p. 27]). *L is* LP *at $X^+=\infty$ if q is bounded below at $X^+=\infty$.*

Corollary 2 (Friedrichs [3, p. 13]). *L is* LP *at $X^+<\infty$ if q is bounded below at X^+ and $\int_{x_0}^{x}(p(s))^{-1}ds$ is not in $L^2(x_0, X^+)$.*

For the Sears LP criterion let $p=1$ and $\hat{q}=c/x^2$ where c is a constant as yet unspecified. The solutions of $\hat{L}v=0$ are then linear combinations of $v_1=x^{\gamma_1}$ and $v_2=x^{\gamma_2}$ where $\gamma_{1,2}=1/2\pm(c+1/4)^{1/2}$. Clearly $\hat{L}$ is nonoscillatory if $c>-1/4$. If $c\geqq 3/4$ then v_2 is not in $L^2(0, x_0)$ so that $\hat{L}$ is then LP at $X^-=0$. We have thus rederived

Corollary 3 (Sears [4]). *L defined by*

$$Ly = -y'' + qy$$

is LP *at $X^-=0$ if $q-3/4x^2$ is bounded below at $X^-=0$.*

It is interesting to note that the above theorem can be derived from the LP criterion of Weyl in Corollary 1 when use is made of the following Liouville-type transformation:

Lemma. (*See, for example,* [3, p. 16].)

$$Ly = -(py')' + qy, \qquad x_0 < x < X^+,$$

becomes

$$\tilde{L}u = (-d/dt)(\tilde{p}(du/dt)) + \tilde{q}u, \qquad 0 < t < T^+,$$

by the transformation

$$y = vu, \qquad t(x) = \int_{x_0}^{x} v^2(s)ds$$

with v as yet unspecified where

$$\tilde{p} = v^4 p, \qquad \tilde{q} = q - \frac{(pv')'}{v}, \qquad T^+ = \int_{x_0}^{X^+} v^2(s)ds.$$

Furthermore,

$$\int_{x_0}^{X^+} y^2(x)dx = \int_0^{T^+} u^2(t)dt.$$

Thus $\tilde{L}$ *is* LP *at* T^+ *if and only if* L *is* LP *at* X^+.

Now let v in the above Lemma be a solution of the comparison equation

$$\hat{L}v = -(pv')' + \hat{q}v = 0.$$

Then clearly $\tilde{q} = q - \hat{q}$.

If $\hat{L}$ is LP at X^+ then one can select v so that $T^+ = \infty$.

If $\hat{L}$ is nonoscillatory at X^+ then $\tilde{p} = v^4 p > 0$.

Thus the hypothesis of the theorem insures that one can apply the LP criterion of Weyl to conclude that $\tilde{L}$ is LP at $T^+ = \infty$. Hence by the above Lemma L is LP at X^+. Q.E.D.

One can generalize Corollaries 2 and 3 by introducing a new independent variable

$$t(x) = \int^{x} ds/p(s) \tag{2}$$

and setting

$$\hat{q}(x) = h(t)/p(x), \tag{3}$$

where the function $h(t)$ is as yet unspecified. Then

$$p(x)\hat{L}v = -d^2v/dt^2 + h(t)v. \tag{4}$$

Each $h(t)$ for which $\hat{L}$ is nonoscillatory and LP at X^+ or X^- then leads, via the above theorem, to an LP criterion for L. For example, if one selects

$$h(t) = C/t^2 \tag{5}$$

then $\hat{L}v=0$ has solutions t^{γ_1} and t^{γ_2} where γ_1 and γ_2 are the values indicated in the above discussion of Corollary 3. One thus concludes the following:

COROLLARY 4. *L defined in* (1) *is* LP *at X^+ (X^-) if $C \geqq 1/4$ is such that* (i) *at least one of t^{γ_1} and t^{γ_2} is not in $L^2(x_0, X^+)$ ($L^2(x_0, X^-)$) and* (ii) *$q(x)-C/p(x)t^2$ is bounded below at X^+ (X^-) where t is defined in* (2).

Corollary 2 corresponds to the selection $C=0$. Then $\gamma_1=1$ and $\gamma_2=0$ so that condition (i) becomes simply that t is not in L^2.

Corollary 3 corresponds to the selection $p(x)=1$. This results in $t(x)=x$ and the optimum value of C is seen to be $3/4$.

As a further example of Corollary 4 let $p(x)=x^\alpha$ where α is a real constant. (Corollary 2 will not be applicable to this case if $\alpha<3/2$ since then $t(x)$ will be in L^2.) From (2), (3) and (5)

$$\hat{q}(x) = K/x^{2-\alpha}$$

where $K=(1-\alpha)^2C$. It is then easily seen that $\hat{L}$ is nonoscillatory and LP at $X^-=0$ if $K \geqq (3-2\alpha)/4$. This then proves

COROLLARY 5. *L defined by* (1) *is* LP *at $X^-=0$ if $p=x^\alpha$, $\alpha<3/2$ and $q-(3-2\alpha)/4x^{2-\alpha}$ is bounded below at $X^-=0$.*

Corollary 3 corresponds to the special case $\alpha=0$.

In order to facilitate comparison with Corollary 2 one can rewrite Corollary 5 by replacing x by X^+-x and $X^-=0$ by $X^+<\infty$.

A selection of $h(t)$ in (3) more general than (5), and yet which is readily dealt with, is one for which $\hat{L}v=0$ has a regular singularity at T where $T=\lim t(x)$ as $x \to X^-$. Thus if $T=0$ let

$$h(t) = t^{-2}(C + C_1t + \cdots + C_kt^k)$$

while if $T=\infty$ let

$$h(t) = t^{-2}(C + C_1t^{-1} + \cdots + C_kt^{-k}).$$

In either case, as long as $\gamma_1-\gamma_2$ is not an integer, t^{γ_1} and t^{γ_2} are now the leading terms of two solutions of $\hat{L}v=0$. Hence in Corollary 4 one can replace (ii) by (ii)′ $q(x)-h(t)/p(x)$ is bounded below at X^+ (X^-) where C is described in Corollary 4 but $C_1, C_2, \cdots, C_k$ are unrestricted.

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