1. Introduction. By a clan we mean a compact connected semigroup with unit. We speak of a topological space $X$ supporting a group structure if there is a topological group having $X$ as its underlying space. Hudson and Mostert [7] have proven that a finite dimensional homogeneous clan is a group. There exist [7, p. 41] infinite dimensional homogeneous clans which are not groups; however, certain homogeneous clans (finite or infinite dimensional), namely those whose spaces will support group structures, are shown here to be groups. Thus the question, "Which clans are groups?", has a purely topological answer.

2. Preliminary results. Homomorphisms of topological groups and local groups are assumed to be continuous and open. A monomorphism is a one-to-one homomorphism, and an onto monomorphism is an isomorphism.

For details on the structure of the cohomology groups used here, we refer to a paper of Hu [4]. We denote the additive group of real numbers by $R$ and use only $R$ as a coefficient group. Let $H^n(X)$ represent the usual $n$-dimensional Alexander-Wallace-Spanier cohomology group of the space $X$. We now sketch the structure and give some properties of a cohomology group of a local group [4, p. 415].

Let $V$ be any local group [8, p. 83] and $V'$ a local subgroup of $V$ [8, p. 84]. For each integer $n > 0$, an $n$-function of $V$ modulo $V'$ is a real-valued continuous function $\phi$ defined on a subset $W_n$ of the space $V^n$ for some open neighborhood $W$ of the unit $e$ in $V$ such that

$$\phi(gv_1h_1, \ldots, gv_nh_n) = \phi(v_1, \ldots, v_n)$$

for each $(v_1, \ldots, v_n)$ in $W^n$ and arbitrary $g$ in $V'$ and $h_i$ in $V'$ ($i = 1, 2, \ldots, n$) provided $gv_ih_i$ is defined and is in $W$ for each $i = 1, 2, \ldots, n$.

Two $n$-functions, $\phi: W^n \to R$, of $V$ modulo $V'$ are said to be equivalent ($\phi \equiv \psi$) if there exists an open neighborhood $W''$ of $e$ contained in the intersection $W \cap W''$ such that $\phi$ agrees with $\psi$ on $W''$. Thus
the $n$-functions of $V$ modulo $V'$ are divided into disjoint equivalence classes, called the $n$-cochains of $V$ modulo $V'$. Denote by $C^n(V, V')$ the set of all $n$-cochains of $V$ modulo $V'$. In a straightforward manner one can define an addition on $C^n(V, V')$ so that it becomes an abelian group. In case $n = 0$, set $C^0(V, V') = \mathbb{R}$.

Define a coboundary operator $\delta: C^n(V, V') \to C^{n+1}(V, V')$ by the following procedure. If $c$ is in $C^n(V, V')$, let $\delta(c) = 0$. For $n > 0$ and $c$ in $C^n(V, V')$, choose an $n$-function, $\phi: W \to \mathbb{R}$, of $V$ modulo $V'$ such that $c = [\phi]$, that is, $\phi$ is an element of the equivalence class $c$. Choose an open neighborhood $W_0$ of $e$ in $V$ such that if $u$ and $v$ are in $W_0$, then $u^{-1}v$ is defined and is in $W$. Define an $(n+1)$-function $\phi: W \to \mathbb{R}$ by setting

$$\delta(\phi)(v_1, \ldots, v_{n+1}) = \phi(v_1^{-1}v_2, \ldots, v_{n}^{-1}v_{n+1}) + \sum_{i=1}^{n+1} (-1)^i \phi(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n+1})$$

for each $(v_1, \ldots, v_{n+1})$ in $W_0^{n+1}$. It can be seen that $\delta(\phi)$ is an $(n+1)$-function of $V$ modulo $V'$ and that $[\delta(\phi)]$ does not depend on the choice of $\phi$. Then we can define $\delta(c) = [\delta(\phi)]$. Using the coboundary operator and $C^n(V, V')$, we construct the $n$th cohomology group of $V$ modulo $V'$ and denote this by $H^n(V, V')$.

Let $G$ be a compact connected group, and choose $V$ to be a neighborhood of the unit $e$ in $G$. Considering $G$ as operating on itself by left translations, it follows from results of Hu [4] that there exists an isomorphism $k: H^n(G) \to H^n(V, \{e\})$.

Let $G$ and $L$ be compact connected groups and $f: G \to L$ an onto homomorphism. Let $W$ be an open neighborhood of the unit $e_2$ in $L$ and $V = f^{-1}(W)$. Let $g$ be the restriction of $f$ to $V$, that is, $g = f|_V$. We want to define a homomorphism $g^*$ from $C^n(W, \{e_2\})$ to $C^n(V, \{e_1\})$. If $n = 0$, we let $g^*$ be the identity on $R$. Assume that $n > 0$ and choose $c$ in $C^n(W, \{e_2\})$. Choose an $n$-function, $\phi: W \to \mathbb{R}$, of $W$ modulo $\{e_2\}$ such that $c = [\phi]$. Define an $n$-function, $g^*(\phi): [g^{-1}(W)]^n \to R$, of $V$ by setting $g^*(\phi)(v_1, \ldots, v_n) = \phi(g(v_1), \ldots, g(v_n))$ for each $(v_1, \ldots, v_n)$ in $[g^{-1}(W)]^n$. If $\phi$ and $\psi$ are equivalent $n$-functions of $W$, then $\phi$ agrees with $\psi$ on some $U^n$ where $U$ is an open neighborhood of $e_2$. If this is the case, then $g^*(\phi)$ agrees with $g^*(\psi)$ on $[g^{-1}(U)]^n$ and this open neighborhood of $e_1$ has the properties necessary to insure that $g^*(\phi)$ and $g^*(\psi)$ are equivalent. Thus we define $g^*(c) = [g^*(\phi)]$. It is clear that $g$ is a homomorphism which commutes with the coboundary operator and hence induces a unique homomorphism $g^*: H^n(W, \{e_2\}) \to H^n(V, \{e_1\})$. 

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Lemma 1. If \( G \) and \( L \) are compact connected Lie groups and \( f: G \to L \) is an onto homomorphism, then for each integer \( n \geq 0 \), \( f^*: H^n(L) \to H^n(G) \) is a monomorphism.

Proof. Choose any open neighborhood \( W \) of the unit \( e' \) in \( L \) and let \( V = f^{-1}(W) \). For any integer \( n \geq 0 \), let \( k: H^n(G) \to H^n(V, \{ e \}) \) (\( e \) is the unit of \( G \)) and \( k': H^n(L) \to H^n(W, \{ e' \}) \) be the isomorphism described by Hu [4]. Letting \( g = f|_V \) and \( g^*: H^n(W, \{ e' \}) \to H^n(V, \{ e \}) \) be the homomorphism induced by \( g \), it is straightforward to see that \( g^*k' = kf^* \). Thus to show that \( f^* \) is a monomorphism, it suffices to show that there exist neighborhoods \( V \) of the unit in \( G \) and \( W \) of the unit in \( L \) such that \( (f|_V)^* \) is a monomorphism. To accomplish this we use the results and terminology of Pontrjagin [8, Chapter IX]. Let \( U' \) be any open neighborhood of \( e' \) and \( U = f^{-1}(U') \). Considering \( U \) and \( U' \) as local Lie groups, there correspond infinitesimal groups \( S \) to \( U \) and \( S' \) to \( U' \). Let \( \phi: S \to S' \) be the homomorphism corresponding to \( f|_U \). Denote the kernel of \( \phi \) by \( K \). \( K \) is normal in \( S \) and hence there exists a normal subgroup \( T \) of \( S \) such that \( K + T = S \) and \( K \cap T = \{ 0 \} \). Define \( \lambda: T \to S \) by \( \lambda(x) = x \) for each \( x \) in \( T \). Then \( \lambda \) is one-to-one and onto. Let \( M \) be the local subgroup of \( U \) corresponding to \( T \) and \( h: M \to U \) the homomorphism corresponding to \( \lambda \). Then \( (f|_U)h \) is a local isomorphism, that is, there exist open neighborhoods \( V_1 \) of the unit in \( M \) and \( W \) of the unit in \( U' \) such that \( ((f|_U)h)|_{V_1}: V_1 \to W \) is an onto monomorphism. Let \( V = f^{-1}(W) \) and \( g = h|_{V_1} \). Then \( (f|_V)g: V_1 \to W \) induces
\[
((f|_V)g)^*: H^n(W, \{ e' \}) \to H^n(V, \{ e' \}).
\]
It is straightforward to verify that \( (f|_V)g^* \) is a monomorphism and that \( (f|_V)g^* = g^*(f|_V)^* \). Therefore \( (f|_V)^* \) must be a monomorphism.

The referee has pointed out that a much shorter proof of Lemma 1 exists; however, the proof given here generalizes easily to a proof for Theorem 2.

Lemma 2. Let \( G \) be a compact connected Lie group of dimension \( n \). If \( A \) is a closed proper subset of \( G \) and \( i: A \to G \) is the inclusion function, then the induced homomorphism \( i^*: H^n(G) \to H^n(A) \) is not a monomorphism.

Proof. Choose \( x \) in \( G \setminus A \) and an open neighborhood \( V \) of \( x \) which is homeomorphic to \( R^n \) and such that \( \text{cl}(V) \cap A = \phi \). Set \( B = \text{cl}(G \setminus V) \). Let \( j: (G, \phi) \to (G, B) \) and \( k: (B, \phi) \to (G, \phi) \) be inclusion functions. It is known [2, p. 314] that \( j^* \) maps \( H^n(G, B) \) onto \( H^n(G, \phi) \). Looking at the exact sequence
one sees that $k^*$ is the zero function. Let $t: A \rightarrow B$ be the inclusion function. Then $kt = i$ implies that $t^*k^* = i^*$ and hence that $i^*$ is the zero function. Since $H^n(G) \neq \{0\}$ [3], it follows that $i^*: H^n(G) \rightarrow H^n(A)$ is not a monomorphism.

3. **Main result.** The proof of the following uses results about compact connected semigroups which can be found in a paper by A. D. Wallace [9].

**Theorem 1.** If $(S, \cdot)$ is a clan and $S$ will support a group structure, then $(S, \cdot)$ is a group.

**Proof.** Since $S$ will support a group structure, it follows [6, pp. 172–179] that there exists an inverse system of compact connected Lie groups, $(G_\alpha, \pi_{\alpha\beta}, D)$, such that $\lim_{\alpha} G_\alpha = S$ and each $\pi_{\alpha\beta}$ is an onto homomorphism. Let $e$ be idempotent in the minimal ideal of $S$ and $i:e \cdot S \cdot e \rightarrow S$ be the injection function. For each $\alpha$ in $D$, let $\pi_\alpha: S \rightarrow G_\alpha$ be the projection function $h_\alpha = \pi_\alpha| e \cdot S \cdot e$, and $i_\alpha: \pi_\alpha(e \cdot S \cdot e) \rightarrow G_\alpha$ be the injection function. Clearly, we have $\pi_\alpha i(x) = i_\alpha h_\alpha(x)$ for each $x$ in $e \cdot S \cdot e$.

Suppose that $\pi_\alpha(e \cdot S \cdot e) \neq G_\alpha$ for some $\alpha$ in $D$. By the continuity axiom, $(H^n(G_\alpha), \pi_{\alpha\beta}^*, D)$ is a direct system of groups for each integer $n \geq 0$. By Lemma 1, each $\pi_{\alpha\beta}^*$ is a monomorphism and thus each $\pi_\alpha^*$ is a monomorphism. For $p$, the dimension of $G_\alpha$,

$$h_\alpha^*: H^p(G_\alpha) \rightarrow H^p(e \cdot S \cdot e)$$

is not a monomorphism, but

$$i_\alpha^* \pi_\alpha^*: H^p(G_\alpha) \rightarrow H^p(e \cdot S \cdot e)$$

is a monomorphism. This is a contradiction since $h_\alpha^* i_\alpha^* = i^* \pi_\alpha^*$. Thus for each $\alpha$ in $D$, $\pi_\alpha(e \cdot S \cdot e) = G_\alpha$ and $(S, \cdot)$ is a group since $(e \cdot S \cdot e, \cdot)$ is a group.

4. **Further remarks.** Theorem 1 is actually a corollary to a more general theorem. We state, with only an indication of proof, this more general theorem because we believe that a much better result can be obtained.

If $H$ is a closed subgroup of a compact connected group $G$, then the left cosets of $H$ can be considered as the points of new space, called the coset space $G/H$, with the customary quotient topology. Choose an inverse system of compact connected Lie groups, $(G_\alpha, \pi_{\alpha\beta}, D)$
such that \( \lim_{\alpha} G_{\alpha} = G \) and such that each \( \pi_{\alpha \beta} \) is an onto homomorphism. Setting \( H_{\alpha} = \pi_{\alpha}(H) \) and \( \pi_{\alpha \beta}(g H_{\alpha}) = \pi_{\alpha \beta}(g) H_{\beta} \), it can be seen \([1]\) that \( (G_{\alpha}/H_{\alpha}, \pi_{\alpha \beta}, D) \) is an inverse system and that the \( \lim_{\alpha} G_{\alpha}/H_{\alpha} = G/H \). For \( \alpha \) in \( D \), we say that \( G_{\alpha}/H_{\alpha} \) is admissible if \( H_n(G_{\alpha}/H_{\alpha}) \neq \{0\} \) where the dimension of \( G_{\alpha}/H_{\alpha} \) is \( n \). Associated with each of the homomorphisms \( \pi_{\alpha \beta} : G_{\alpha} \rightarrow G_{\beta} \), there are neighborhoods \( V_1 \) of the unit in \( G_{\alpha} \), \( V_2 \) of the unit in the kernel of \( \pi_{\alpha \beta} \), and \( V_3 \) of the unit in a local Lie subgroup of \( G_{\beta} \). We shall say that \( \pi_{\alpha \beta} \) induced by \( \pi_{\alpha \beta} \) is admissible if \( V_1 \), \( V_2 \), and \( V_3 \) can be chosen such that \( V_3 \) is a neighborhood of the unit in \( H_{\beta} \). Further, we say that the coset space is admissible if it is the inverse limit of an inverse system, \( (G_{\alpha}/H_{\alpha}, \pi_{\alpha \beta}', D) \), where each factor space and each function \( \pi_{\alpha \beta}' \) are admissible.

**Theorem 2.** If \( (S, \cdot) \) is a clan and \( S \) is an admissible coset space, then \( (S, \cdot) \) is a group.

The proof of the above, with the requirement that \( S = G/H \) with \( H \) connected can be accomplished in the same manner as the proof of Theorem 1. The requirement that \( H \) be connected can be removed by using the results of Hu \([5]\).

We suspect that the hypothesis of Theorem 2 can be weakened to require only that \( S \) be a coset space.

Note that Theorem 1 provides a tool for conveniently showing that certain spaces cannot support group structures. For example, let \( S \) be the Tychonoff cube and \( G \) the circle group. \( S \) will support a clan which is not a group. Thus \( S \times G \) with the coordinatewise operation is a clan which is not a group. Thus \( S \times G \), which is homogeneous and does not have fixed point property, cannot support a group structure. Indeed, the product of a continuum which will support a clan that is not a group with any continuum which will support a clan cannot support a group structure.

**References**


University of Kentucky