GROWTH ESTIMATES OF CONVEX FUNCTIONS

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In this paper we generalize a result of Loewner's by use of an iterated form of the Schwarz Lemma. This tool is used repeatedly to obtain several results on the growth of convex analytic functions.

Preliminaries. By a convex function we will mean a univalent analytic function, defined on the unit disc, whose range is a convex point set. Such a function f satisfies the differential inequality:

$$\mathrm{Re}\{zf''/f'\} > -1 \quad \text{for all } z \text{ in the unit disc.}$$

By a starlike function we will mean a univalent analytic function, defined on the unit disc, whose range is a starlike set with respect to the image of the origin. That is, if ξ is the image of 0, and w lies within the range, then the line segment joining ξ and w lies within the range as well. If we assume that $f(0)=0$, such a function satisfies the differential inequality:

$$\mathrm{Re}\{zf'/f\} > 0 \quad \text{for all } z \text{ in the unit disc.}$$

A trivial consequence of the above is that a function f is convex if and only if zf' is starlike. As a result, all theorems on convex functions will yield results on starlike functions, and conversely.

In what follows, we shall assume that the functions under consideration are normalized. That is, $f(0)=0$, $f'(0)=1$. This is no essential restriction, for if $g(z)$ is any univalent function defined on the unit disc, then

$$f(z) = (g(z) - g(0))/g'(0)$$

is a normalized, univalent function. We shall for simplicity occasionally assume in addition that $f''(0) \geqq 0$. Again, this is no restriction, for by appropriate choice of θ, $f(z)=e^{-i\theta}g(e^{i\theta}z)$ will have a real and nonnegative second derivative at 0.

Loewner [2] proved that for a convex function $f(z)=z+\cdots$, $|f'(z)| \leqq 1/(1-r)^2$, $|z|=r$. Theorem 1 improves upon this result to give growth estimates of f' involving the second coefficient in the power series expansion of f. First we state and prove an iterated form of Schwarz' Lemma. The method of proof goes back at least as far as Landau [1, p. 307]. The lemma itself appeared in [3].

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LEMMA 1. *Let* $f(z)=a_1z+\cdots$ *be an analytic map of the unit disc into itself. Then* $|a_1|\leqq 1$, *and*

$$|f(z)|\leqq r(r+|a_1|)/(1+|a_1|r).$$

Equality holds at some $z\neq 0$ *if and only if*

$$f(z)=\frac{e^{-it}z(z+a_1e^{it})}{1+\bar{a}_1e^{-it}z,}\, t\geqq 0.$$

PROOF. That $|a_1|\leqq 1$ is an immediate consequence of Schwarz' Lemma. We mention it so that the inequality here becomes an improvement on Schwarz' Lemma, in that the estimate is better when $|a_1|<1$.

Let $g(z)=f(z)/z=a_1+a_2z+\cdots$. Then the maximum principle shows that $|g(z)|<1$ in the unit disc.

Let $h(z)=(g(z)-a_1)/(1-\bar{a}_1g(z))=b_1z+\cdots$. Then as $|g|<1$, also $|h|<1$. Further, $h(0)=0$, and thus, by the familiar form of Schwarz' Lemma,

$$|h(z)|\leqq r.$$

Equality holds for some $z\neq 0$ if and only if $h(z)=e^{it}z$. But

$$|f(z)|=\left|z\frac{h(z)+a_1}{1+\bar{a}_1h(z)}\right|\leqq r\frac{|h(z)|+|a_1|}{1+|a_1h(z)|}\leqq r\left(\frac{r+|a_1|}{1+|a_1|r}\right).$$

Further, it is clear that equality can hold at a point z only if $|h(z)|=|z|$, from which it follows that

$$f(z)=\frac{e^{it}z(z+a_1e^{it})}{1+\bar{a}_1ze^{it}.}$$

THEOREM 1. *Let* $f(z)=z+a_2z^2+\cdots$ *be convex in the unit disc. Then*

$$|f'(z)|\leqq\left(\frac{1}{1-r^2}\right)\left[\frac{1+r}{1-r}\right]^{\rho},\tag{1}$$

where $|a_2|=\rho$. *Equality holds at some* $z\neq 0$ *if and only ij*

$$f(z)=\frac{e^{-it}}{2\rho}\left(\left[\frac{1+e^{it}z}{1-e^{it}z}\right]^{\rho}-1\right).\tag{2}$$

This function (2) *maps the unit disc onto a sector of opening angle* $\rho\pi$, *the apex of the sector lying at a point of modulus* $1/2\rho$, *and* 0 *lying on the axis of symmetry.*

PROOF. That (2) maps the unit disc as indicated above can be seen through the Schwarz-Christoffel representation for mappings onto polygons.[1]

$$f'(z) = (1 + e^{it}z)^{\rho-1}(1 - e^{it}z)^{-\rho-1}.$$

Consequently, f maps onto a 2-sided polygon (sector) whose exterior angle at the apex is $(1-\rho)\pi$. Hence the interior angle is $\rho\pi$. The remaining details of this mapping are easily verified.

We can, as before, perform a rotation in the z- and w-planes so that $a_2=\rho>0$. So $f(z)=z+\rho z^2+a_3z^3+\cdots$. Let $\psi(r)=(1-r)^{1+\rho}(1+r)^{1-\rho}$. Let

$$\phi(z) = \log |f'(z)| + \log \psi(r).$$

We shall show that $\phi(z)\leqq 0$. In fact, we shall show even more, that

$$\partial\phi/\partial r \leqq 0 \quad \text{for } z \neq 0.$$

Let us note that $\phi(0)=0$, and that ϕ is continuous in z. Now

$$\partial\phi/\partial r = (\partial/\partial r) \log |f'(z)| - (1 + \rho)/(1 - r) + (1 - \rho)/(1 + r).$$

But

$$r(\partial/\partial r) \log |f'(z)| = \text{Re}\,\{zf''/f'\},$$

so

$$r(\partial\phi/\partial r) = \text{Re}\{zf''/f'\} - 2r(r + \rho)/(1 - r^2). \tag{3}$$

Now, if $h(z)=zf''/f'$, we have $1+h(z)$ is a function of positive real part, or $1+h=(1+g)/(1-g)$, for $|g|<1$, $g(z)=\rho z+\cdots$,

$$h = 2g/(1 - g).$$

By Lemma 1, $|g(z)|\leqq r(r+\rho)/(1+r\rho)$, and thus

$$|h(z)| \leqq 2r(r + \rho)/(1 - r^2).$$

Then

$$r(\partial\phi/\partial r) = \text{Re}\{zf''/f'\} - 2r(r + \rho)/(1 - r^2),$$

and

$$\text{Re}\,\{zf''/f'\} \leqq |zf''/f'| = |h| \leqq 2r(r + \rho)/(1 - r^2).$$

Thus by (3)

$$r(\partial\phi/\partial r) \leqq 0.$$

This completes the proof that

[1] See, for example, Hille, *Analytic function theory*. II, Ginn, Boston, Mass., p. 372.

$$|f'(z)| \leqq \frac{1}{1-r^2}\left[\frac{1+r}{1-r}\right]^\rho$$

If equality is attained at a point, an investigation of the above proof shows that $h(z)$ must achieve the estimate for it along a ray. Then, as in Lemma 1, h is determined up to a rotational constant:

$$h(z) = 2e^{it}z(z+\rho)/(1-z^2).$$

From this it follows that

$$f(z) = (e^{-it}/2\rho)([(1+e^{it}z)/(1-e^{it}z)]^\rho - 1). \tag{4}$$

We observe the following well-known[2] result:

COROLLARY 1.1. *Let* $f(z)=z+a_3z^3+a_5z^5+\cdots$ *be an odd convex function. Then*

$$|f'(z)| \leqq 1/(1-r^2).$$

PROOF. An odd function has vanishing even coefficients, and on setting $a_2=\rho=0$ in (1) we obtain the result.

COROLLARY 1.2. *Let* $f(z)=z+\rho z^2+a_3z^3+\cdots$ *be convex in the unit disc. Then*

$$|f(z)| \leqq (1/2\rho)([(1+r)/(1-r)]^\rho - 1).$$

This estimate is sharp.

PROOF. As we remarked in the preliminaries, it is no loss of generaity to assume as we have here that $a_2=\rho\geqq 0$. The result follows on integration of (1). The same function (4) is extremal for both Theorem 1 and this corollary.

COROLLARY 1.3. *Let* $f(z)=z+2\rho z^2+a_3z^3+\cdots$ *be starlike in the unit disc,* $0\leqq\rho\leqq 1$. *Then*

$$|f(z)| \leqq \frac{r}{1-r^2}\left[\frac{1+r}{1-r}\right]^\rho .$$

PROOF. We observed earlier that g is convex if and only if $zg'(z)$ is starlike. Or, what is the same, f is starlike if and only if $\int_0^z (f(z)/z)dz$ is convex. Hence

$$\int_0^z (f(z)/z)dz = z + \rho z^2 + \frac{a_3}{3}z^3 + \cdots$$

is convex, and, by Theorem 1,

[2] See, for example, Nehari, *Conformal mapping*, McGraw-Hill, New York, p. 238.

$$|f(z)/z| \leqq \frac{1}{1-r^2}\left[\frac{1+r}{1-r}\right]^\rho.$$

The result then follows. The function

$$(z/(1-z^2))[(1+z)/(1-z)]^\rho$$

is starlike and achieves this estimate. To determine the range of this function, we again use the Schwarz-Christoffel representation. Computing the derivative of this function,

$$\begin{aligned} f'(z) &= \frac{1+2\rho z+z^2}{(1-z)^{2+\rho}(1+z)^{2-\rho}} \\ &= \frac{(1+e^{i\theta}z)(1+e^{-i\theta}z)}{(1-z)^{2+\rho}(1+z)^{2-\rho}}, \qquad e^{i\theta}+e^{-i\theta}=2\rho. \end{aligned}$$

The range of f is thus a 4-sided polygon with exterior angles $(2+\rho)\pi$, $-\pi$, $(2-\rho)\pi$, $-\pi$. It is then easily seen that this polygon is the entire plane minus two radial slits extending to ∞. The two slits make an angle of $(1-\rho)\pi$ with one another.

THEOREM 2. *Let* $f(z)=z+\rho z^2+a_3z^3+\cdots$ *be convex in the unit disc,* $\rho\geqq 0$. *Then*

$$|f'(z)| \geqq 1/(1+2\rho r+r^2).$$

PROOF.

$$r(\partial/\partial r)\log|f'(z)| = \operatorname{Re}\{zf''/f'\}. \tag{5}$$

Now $1+zf''/f'=(1+g)/(1-g)$, for some $|g(z)|<1$, and

$$\begin{aligned} 1+\operatorname{Re}\left\{\frac{zf''}{f'}\right\} &= \operatorname{Re}\left\{\frac{1+g}{1-g}\right\} = \frac{1-g\bar{g}}{|1-g|^2} \\ &\geqq \frac{1-g\bar{g}}{(1+|g|)^2} = \frac{1-|g|}{1+|g|}. \end{aligned}$$

Hence

$$\operatorname{Re}\{zf''/f'\} \geqq (-2|g|)/(1+|g|). \tag{6}$$

Applying Lemma 1 to g,

$$|g(z)| \leqq r(r+\rho)/(1+\rho r).$$

Hence from (6) we have

$$\operatorname{Re}\{zf''/f'\} \geqq \frac{-2r(r+\rho)/(1+\rho r)}{1+r(r+\rho)/(1+\rho r)} = \frac{-2r(r+\rho)}{1+2\rho r+r^2}.$$

But then by (5),

$$(\partial/\partial r)\log|f'(z)| \geqq -2(r+\rho)/(1+2\rho r+r^2). \tag{7}$$

On integrating (7) with respect to r, and then exponentiating, we obtain

$$|f'(z)| \geqq 1/(1+2\rho r+r^2).$$

If $\rho<1$, the function

$$f(z) = \frac{1}{2i\sin\theta}\log\left(\frac{1-e^{-i\theta}z}{1-e^{i\theta}z}\right), \qquad \rho=\cos\theta$$

is convex and achieves the estimate of the theorem along the negative real axis from 0 to -1. By considering the Schwarz-Christoffel representation, it is seen that this function maps the unit disc onto an infinite strip, of width $\pi/(2\sin\theta)$, with the image of the origin lying at distances from the two parallel boundary lines $(\pi-\theta)/(2\sin\theta)$, $\theta/(2\sin\theta)$ respectively.

If $\rho=1$, the function $f(z)=z/(1-z)$ achieves the estimate of the theorem along the negative real axis from 0 to -1. This function maps the unit disc onto a half plane.

COROLLARY 2.1. *Let $f(z)=z+\rho z^2+a_3z^3+\cdots$ be convex in the unit disc. Then*

$$|f(z)| \geqq \frac{1}{(1-\rho^2)^{1/2}}\tan^{-1}\left[\frac{r+\rho}{(1-\rho^2)^{1/2}}\right], \qquad \rho\geqq 0.$$

PROOF. Let Γ be the straight line segment joining 0 and $f(z)$. Since f is convex, Γ lies in the range of f. Let $\gamma=f^{-1}(\Gamma)=\{z: f(z)\in\Gamma\}$. Then γ is an arc in the unit disc joining 0 and z. It follows that

$$|f(z)| = \int_\gamma |f'(z)||dz| \geqq \int_0^r |f'(s)|\,ds \geqq \int_0^r \frac{ds}{1+2s\rho+s^2}.$$

The stated result then follows.

Now consider the function

$$f(z) = \frac{1}{2i\sin\theta}\log\left(\frac{1-e^{-i\theta}z}{1-e^{i\theta}z}\right) = \int_0^z \frac{d\xi}{(1-e^{i\theta}\xi)(1-e^{-i\theta}\xi)},$$

which was discussed in the preceding theorem. For $e^{i\theta}+e^{-i\theta}=2\rho$, it is clear that $f'(r)=(1+2\rho r+r^2)^{-1}$, which is the lower estimate for $|f'(z)|$. Hence in our proof above, if Γ is the straight line segment joining 0 and $f(r)$, then γ lies in the positive real axis. Hence all the inequalities can be replaced by equalities for this function, which shows the estimate to be sharp.

COROLLARY 2.2. *Let* $f(z)=z+\rho z^2+a_3z^3+\cdots$ *be starlike in the unit disc. Then*

$$|f(z)| \geqq r/(1+r\rho+r^2).$$

PROOF. The proof is exactly the same as the proof of Corollary 1.3. The function which shows the estimate to be sharp is $z/(1+\rho z+z^2)$.

THEOREM 3. *Let* $f(z)=z+\rho z^2+a_3z^3+\cdots$ *be convex in the unit disc. Then*

$$|f''(z)| \leqq \frac{2(r+\rho)}{(1-r^2)^2}\left[\frac{1+r}{1-r}\right]^\rho.$$

This result is sharp.

PROOF. As in the proof of Theorem 2, there exists a function g, $|g|<1$, $g(z)=\rho z+\cdots$, so that

$$zf''/f' = 2g/(1-g).$$

$$\left|\frac{zf''}{f'}\right| = \left|\frac{2g}{1-g}\right| \leqq \frac{2|g|}{1-|g|} \leqq \frac{2r(r+\rho)}{1-r^2},$$

on applying Lemma 1 to g. Hence

$$|f''| \leqq \frac{2r(r+\rho)}{1-r^2}\left|\frac{f'}{r}\right| \leqq \frac{2(r+\rho)}{(1-r^2)^2}\left[\frac{1+r}{1-r}\right]^\rho$$

when we apply the estimate of Theorem 1 for $|f'|$. Again, the extremal function of Theorem 1 achieves this estimate.

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