A NOTE ON BERNSTEIN POLYNOMIAL TYPE APPROXIMATIONS

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The Bernstein polynomial of order n for a function f defined on $[0, 1]$ is defined by

$$B_n(f(t)) = \sum_{m=0}^{n} \binom{n}{m} t^m(1-t)^{n-m} f\left(\frac{m}{n}\right) = \sum_{m=0}^{n} \binom{n}{m} [\Delta^{n-m}t^m] f\left(\frac{m}{n}\right)$$

where

$$\Delta^{n-m}t^m = \sum_{\nu=0}^{n-m} \binom{n-m}{\nu} (-1)^{n-m-\nu} t^{n-\nu}.$$

These polynomials provide a method for the uniform approximation of a function f which is continuous on the interval $[0, 1]$ and has values in a Banach space X by use of the function sequence $\{t^i\}_{i=0}^{\infty}$. Given a sequence $\Phi = \{\phi_i\}_{i=0}^{\infty}$ of continuous functions from $[0, 1]$ into $B[X, Y]$, the space of bounded linear transformations from a Banach space X into a Banach space Y, we define

$$B_n(\Phi, f) = \sum_{m=0}^{n} \binom{n}{m} [\Delta^{n-m}\phi_m] f\left(\frac{m}{n}\right)$$

to be the Φ-Bernstein approximation of f of order n. In this note we shall consider the question of uniform convergence of such approximations.

DEFINITION. The sequence Φ is said to satisfy condition A if there exists an $M > 0$ such that if $\{x_m\}_{m=0}^{\infty}$ is a bounded sequence in X, then

$$\left\| \sum_{m=0}^{n} \binom{n}{m} [\Delta^{n-m}\phi_m] \cdot x_m \right\|_{C(Y)} \leqq M \sup_{(m)} \|x_m\| \qquad \text{for all } n \geqq 0;$$

where $\| \; \|_{C(Y)}$ denotes the uniform norm on the function space $C(Y)$ of continuous functions from $[0, 1]$ into Y.

THEOREM. *The following two statements are equivalent:*
(1) $B_n(\Phi, f)$ *converges in* $C(Y)$ *for each* f *in* $C(X)$.
(2) Φ *satisfies condition* A.

PROOF. We note first that condition A identifies Φ as a Hausdorff moment sequence from X into $C(Y)$ [**1**, Lemma 8]. (Reference 8 in

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[1] is reference [2] of this paper.)

Suppose (2) holds, i.e., Φ is a Hausdorff moment sequence from X into $C(Y)$. By the proof of the first part of [1, Theorem 3], there exists a continuous linear transformation T from $C(X)$ into $C(Y)$ such that, for each g in $C(X)$, $T(g)(s)=\int_0^1 d_tK(s,t)\cdot g(t)$ and such that $T(f\cdot x)(s)=\int_0^1 d_tK(s,t)\cdot f(t)\cdot x$ for each f in $C(R)$ and each x in X, where R denotes the real field and such that, furthermore, $\phi_m(s)\cdot x = T(t^m\cdot x)$ for $m=0, 1, 2, \cdots$ and each x in X. We then have that

$$B_n(\Phi, f) = \sum_{m=0}^{n}\binom{n}{m}[\Delta^{n-m}\phi_m]\cdot f\left(\frac{m}{n}\right)$$
$$= T\left[\sum_{m=0}^{n}\binom{n}{m}[\Delta^{n-m}t^m]f\left(\frac{m}{n}\right)\right] = T[B_n(f)]$$

and since T is continuous and $B_n(f)$ converges in norm to f, we have that $B_n(\Phi,f)$ converges in norm to $T(f)=\int_0^1 dK\cdot f$, and hence (1) holds.

Now suppose (1) holds. It is easily seen that for each n,

$$B_n(\Phi, f) = \sum_{m=0}^{n}\binom{n}{m}[\Delta^{n-m}\phi_m]\cdot f\left(\frac{m}{n}\right)$$

defines a continuous linear transformation from $C(X)$ into $C(Y)$ since it is the finite sum of such transformations. Since $\{B_n(\Phi,f)\}_{n=0}^{\infty}$ converges for each f in $C(X)$, we have by the uniform boundedness principle that there exists a constant $M>0$ such that for each f in $C(X)$ and each $n=0, 1, 2, \cdots$,

$$\|B_n(\Phi,f)\|_{C(Y)} = \left\|\sum_{m=0}^{n}\binom{n}{m}[\Delta^{n-m}\phi_m]\cdot f\left(\frac{m}{n}\right)\right\|_{C(Y)}$$
$$\leqq M\|f\|_{C(X)}.$$

Now suppose a bounded sequence $\{x_m\}_{m=0}^{\infty}$ of points in X is given. There exists a polygonal function P in $C(X)$ which has the values $P(m/n)=x_m$ for $m=0, 1, \cdots, n$ and P is defined to be linear otherwise. $\|P\|_{C(X)}=\max\|x_m\|$ where the maximum is taken over $m=0, \cdots, n$. Taking P for f above gives

$$\left\|\sum_{m=0}^{n}\binom{n}{m}[\Delta^{n-m}\phi_m]\cdot x_m\right\|_{C(Y)} \leqq M \max_{0\leqq m\leqq n}\|x_m\| \leqq M\sup_{(m)}\|x_m\|$$

and (2) holds.

Corollary. *For the case in which Y is X, (then each ϕ_m is in $C(B[X, X])$) the following two statements are equivalent*:

(1*) *$B_n(\Phi, f)$ converges to f for each f in $C(X)$.*
(2*) *$\phi_m(t) = t^m$ for $m = 0, 1, \cdots$.*

PROOF. Suppose (1*) holds, then by the theorem, condition A is satisfied and there exists T such that $T(t^m \cdot x) = \phi_m \cdot x$ for each m, but then by (1*) $T(f)(s) = \lim_{n\to\infty} B_n(\Phi, f) = f(s)$ for each f in $C(X)$ and hence $T(t^m \cdot x)(s) = s^m \cdot x = \phi_m(s) \cdot x$ for each x and hence $\phi_m(s) = s^m$ for all m.

The proof that (2*) implies (1*) is a trivial modification of the classical proof that $B_n(f)$ converges to f for each f in $C(R)$.

BIBLIOGRAPHY

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