A NOTE ON BERNSTEIN POLYNOMIAL TYPE APPROXIMATIONS

D. H. TUCKER

The Bernstein polynomial of order \( n \) for a function \( f \) defined on \([0, 1]\) is defined by

\[
B_n(f(t)) = \sum_{m=0}^{n} \binom{n}{m} t^m (1 - t)^{n-m} f\left(\frac{m}{n}\right) = \sum_{m=0}^{n} \binom{n}{m} \Delta^{n-m} f\left(\frac{m}{n}\right)
\]

where

\[
\Delta^{n-m} = \sum_{r=0}^{n-m} \binom{n-m}{r} (-1)^{n-m-r}.
\]

These polynomials provide a method for the uniform approximation of a function \( f \) which is continuous on the interval \([0, 1]\) and has values in a Banach space \( X \) by use of the function sequence \( \{t^i\}_{i=0}^{\infty} \). Given a sequence \( \Phi = \{\phi_i\}_{i=0}^{\infty} \) of continuous functions from \([0, 1]\) into \( B[X, Y] \), the space of bounded linear transformations from a Banach space \( X \) into a Banach space \( Y \), we define

\[
B_n(\Phi, f) = \sum_{m=0}^{n} \binom{n}{m} \Delta^{n-m} \phi_m f\left(\frac{m}{n}\right)
\]

to be the \( \Phi \)-Bernstein approximation of \( f \) of order \( n \). In this note we shall consider the question of uniform convergence of such approximations.

Definition. The sequence \( \Phi \) is said to satisfy condition A if there exists an \( M > 0 \) such that if \( \{x_m\}_{m=0}^{\infty} \) is a bounded sequence in \( X \), then

\[
\left\| \sum_{m=0}^{n} \binom{n}{m} \Delta^{n-m} \phi_m \cdot x_m \right\|_{C(Y)} \leq M \sup_{(m)} \|x_m\| \quad \text{for all } n \geq 0;
\]

where \( \| \cdot \|_{C(Y)} \) denotes the uniform norm on the function space \( C(Y) \) of continuous functions from \([0, 1]\) into \( Y \).

Theorem. The following two statements are equivalent:

1. \( B_n(\Phi, f) \) converges in \( C(Y) \) for each \( f \) in \( C(X) \).
2. \( \Phi \) satisfies condition A.

Proof. We note first that condition A identifies \( \Phi \) as a Hausdorff moment sequence from \( X \) into \( C(Y) \) [1, Lemma 8]. (Reference 8 in

Received by the editors April 14, 1966.

492
Suppose (2) holds, i.e., $\Phi$ is a Hausdorff moment sequence from $X$ into $C(Y)$. By the proof of the first part of [1, Theorem 3], there exists a continuous linear transformation $T$ from $C(X)$ into $C(Y)$ such that, for each $g$ in $C(X)$, $T(g)(s) = \int_0^1 dtK(s, t) \cdot g(t)$ and such that $T(f\cdot x)(s) = \int_0^1 dtK(s, t) \cdot f(t) \cdot x$ for each $f$ in $C(R)$ and each $x$ in $X$, where $R$ denotes the real field and such that, furthermore, $\phi_m(s) \cdot x = T(t^m \cdot x)$ for $m = 0, 1, 2, \cdots$ and each $x$ in $X$. We then have that

$$B_n(\Phi, f) = \sum_{m=0}^{n} \binom{n}{m} [\Delta^{n-m}\phi_m] \cdot f \left( \frac{m}{n} \right)$$

and since $T$ is continuous and $B_n(f)$ converges in norm to $f$, we have that $B_n(\Phi, f)$ converges in norm to $T(f) = \int_0^1 dtK \cdot f$, and hence (1) holds.

Now suppose (1) holds. It is easily seen that for each $n$,

$$B_n(\Phi, f) = \sum_{m=0}^{n} \binom{n}{m} [\Delta^{n-m}\phi_m] \cdot f \left( \frac{m}{n} \right)$$

defines a continuous linear transformation from $C(X)$ into $C(Y)$ since it is the finite sum of such transformations. Since $\{B_n(\Phi, f)\}_{n=0}^{\infty}$ converges for each $f$ in $C(X)$, we have by the uniform boundedness principle that there exists a constant $M > 0$ such that for each $f$ in $C(X)$ and each $n = 0, 1, 2, \cdots$,

$$\|B_n(\Phi, f)\|_{C(Y)} = \left\| \sum_{m=0}^{n} \binom{n}{m} [\Delta^{n-m}\phi_m] \cdot f \left( \frac{m}{n} \right) \right\|_{C(Y)} \leq M\|f\|_{C(X)}.$$

Now suppose a bounded sequence $\{x_m\}_{m=0}^{\infty}$ of points in $X$ is given. There exists a polygonal function $P$ in $C(X)$ which has the values $P(m/n) = x_m$ for $m = 0, 1, \cdots, n$ and $P$ is defined to be linear otherwise. $\|P\|_{C(X)} = \max_{0 \leq m \leq n} \|x_m\|$ where the maximum is taken over $m = 0, \cdots, n$. Taking $P$ for $f$ above gives

$$\left\| \sum_{m=0}^{n} \binom{n}{m} [\Delta^{n-m}\phi_m] \cdot x_m \right\|_{C(Y)} \leq M\max_{0 \leq m \leq n} \|x_m\| \leq M\sup_{(m)} \|x_m\|$$

and (2) holds.

**Corollary.** For the case in which $Y$ is $X$, (then each $\phi_m$ is in $C(B[X, X])$) the following two statements are equivalent:
(1*) \( B_n(\Phi, f) \) converges to \( f \) for each \( f \) in \( C(X) \).

(2*) \( \phi_m(t) = t^m \) for \( m = 0, 1, \ldots \).

Proof. Suppose (1*) holds, then by the theorem, condition A is satisfied and there exists \( T \) such that \( T(t^m \cdot x) = \phi_m(x) \) for each \( m \), but then by (1*) \( T(f)(s) = \lim_{n \to \infty} B_n(\Phi, f) = f(s) \) for each \( f \) in \( C(X) \) and hence \( T(t^m \cdot x)(s) = s^m \cdot x = \phi_m(s) \cdot x \) for each \( x \) and hence \( \phi_m(s) = s^m \) for all \( m \).

The proof that (2*) implies (1*) is a trivial modification of the classical proof that \( B_n(f) \) converges to \( f \) for each \( f \) in \( C(R) \).

Bibliography


The University of Utah