SOME NONLINEAR TAUBERIAN THEOREMS

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In a forthcoming study of the n-body problem I have made use of the nonlinear Tauberian theorems obtained in this note. The symbol $\omega(x)$ represents a positive function, increasing for $x>0$. The symbols f, g, h represent functions which are of class C^2 on $(0, \infty)$.

The basic Theorem 1 is due to Boas [1].

THEOREM 1. *If*

$$f(x) = o(x), \qquad x \to 0+,$$

and

$$f''(x) = \omega(|f'(x)|)O(x^{-1}), \qquad x \to 0+,$$

then

$$f'(x) = o(1), \qquad x \to 0+.$$

My first theorem asserts that the clause "$x\to 0+$" may be replaced by "$x\to\infty$."

THEOREM 2. *If*

$$g(x) = o(x), \qquad x \to \infty,$$

and

$$g''(x) = \omega(|g'(x)|)O(x^{-1}), \qquad x \to \infty,$$

then

$$g'(x) = o(1), \qquad x \to \infty.$$

PROOF. The function $f(x)$ defined by

$$f(x) = x^2 g(1/x) - 2\int_{1/x}^{\infty} (g(u)/u^3)du$$

satisfies the hypotheses of Theorem 1.

THEOREM 3. *If*

$$g(x) \sim x, \qquad x \to \infty,$$

and

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$$g''(x) = \omega(|g'(x)|)O(x^{-1}), \qquad x \to \infty,$$

then

$$g'(x) \sim 1, \qquad x \to \infty.$$

PROOF. The function $h(x) = g(x) - x$ satisfies the conditions of Theorem 2 with $\omega(y)$ replaced by $\omega(y+1)$.

THEOREM 4. *Let a be a number satisfying* $0 < a < 1$, *or* $2 \leqq a < \infty$. *If*

$$h(x) \sim x^a, \qquad x \to \infty,$$

and

$$|h''(x)| \leqq A|h'(x)|^b, \quad b = (2-a)/(1-a), \tag{1}$$

then

$$h'(x) \sim ax^{a-1}, \qquad x \to \infty.$$

PROOF. Let $h(x) = g(x^a)$. By (1),

$$|(a-1)x^{a-2}g'(x^a) + a^2x^{2a-2}g''(x^a)| \leqq B|x^{a-1}g'(x^a)|^{(2-a)/(1-a)},$$

where $B = Aa^{(2-a)/(1-a)}$. Then

$$x|g''(x)| \leqq B|g'(x)|^b + C|g'(x)|,$$

for suitable constants B and C. Therefore $g(x)$ satisfies the conditions of Theorem 3 with

$$\omega(y) = By^b + Cy, \tag{2}$$

and the proof is complete.

Observe that in the excluded case $1 < a < 2$ the number b is negative, so that the function $\omega(y)$ defined by (2) is increasing for sufficiently large y, but *not* all $y > 0$. I have not investigated this case further.

A natural conjecture is to consider the case $a = \infty$ to correspond to

THEOREM 5. *If*

$$h(x) \sim e^x, \qquad x \to \infty,$$

and

$$|h''(x)| \leqq A|h'(x)|$$

then

$$h'(x) \sim e^x, \qquad x \to \infty.$$

PROOF. The function $g(x)$ defined by $g(e^x)=h(x)$ satisfies the conditions of Theorem 3.

At the other end of the spectrum it is possible to make several conjectures corresponding to the open case $a=0$, $b=2$ of Theorem 4. First, if $a\neq 0$ the condition $h(x)\sim x^a$ can be replaced by $h(x)\sim a^{-1}(x^a-1)$ and the conclusion by $h'(x)\sim x^{a-1}$. Letting $a\to 0+$ suggests the conjecture that the conditions $h(x)\sim \log x$ and

$$|h''(x)| \leqq A\,|h'(x)|^2 \tag{3}$$

imply

$$h'(x)\sim 1/x, \qquad x\to\infty. \tag{4}$$

This is made plausible by the fact that $\log x$ itself satisfies (3). However, the example $h(x)=\log x+\epsilon\cos(\log x)$ shows the conjecture to be false.

The following theorem holds and is adequate for the applications.

THEOREM 6. *If*

$$\int^x yh'(y)dy\sim x, \qquad x\to\infty, \tag{5}$$

and (3) *is satisfied, then* $h'(x)\sim 1/x$, $x\to\infty$.

For the proof apply Theorem 3 to the function $g(x)$ defined by the integral appearing in (5).

ADDED IN PROOF (DECEMBER 19, 1966). The conclusion of Theorem 4 is still valid for $1<a<2$ provided (1) is understood to mean

$$|h'(x)|^{-b}|h''(x)| \leqq A, \qquad b=(2-a)/(1-a).$$

This follows by the same argument, but substituting for Theorem 3 a theorem of Karamata (Amer. J. Math. **61** (1939), 769–770).

REFERENCE

1. R. P. Boas, Jr., *A Tauberian theorem connected with the problem of three bodies*, Amer. J. Math. **61** (1939), 161–164.

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