SOME NONLINEAR TAUBERIAN THEOREMS

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In a forthcoming study of the n-body problem I have made use of the nonlinear Tauberian theorems obtained in this note. The symbol $\omega(x)$ represents a positive function, increasing for $x>0$. The symbols $f, g, h$ represent functions which are of class $C^2$ on $(0, \infty)$.

The basic Theorem 1 is due to Boas [1].

Theorem 1. If

$$f(x) = o(x), \quad x \to 0 +,$$

and

$$f''(x) = \omega(\mid f'(x) \mid) O(x^{-1}), \quad x \to 0 +,$$

then

$$f'(x) = o(1), \quad x \to 0 +.$$

My first theorem asserts that the clause "$x \to 0+$" may be replaced by "$x \to \infty$.”

Theorem 2. If

$$g(x) = o(x), \quad x \to \infty,$$

and

$$g''(x) = \omega(\mid g'(x) \mid) O(x^{-1}), \quad x \to \infty,$$

then

$$g'(x) = o(1), \quad x \to \infty.$$

Proof. The function $f(x)$ defined by

$$f(x) = x^2 g(1/x) - 2 \int_{1/x}^{\infty} (g(u)/u^3) du$$

satisfies the hypotheses of Theorem 1.

Theorem 3. If

$$g(x) \sim x, \quad x \to \infty,$$

and

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\[ g''(x) = \omega \left( \left| g'(x) \right| \right) O(x^{-1}), \quad x \to \infty, \]

then
\[ g'(x) \sim 1, \quad x \to \infty. \]

**Proof.** The function \( h(x) = g(x) - x \) satisfies the conditions of Theorem 2 with \( \omega(y) \) replaced by \( \omega(y + 1) \).

**Theorem 4.** Let \( a \) be a number satisfying \( 0 < a < 1 \), or \( 2^a < 1 \). If
\[ h(x) \sim x^a, \quad x \to \infty, \]
and
\[ | h''(x) | \leq A | h'(x) |^b, \quad b = (2 - a)/(1 - a), \]
then
\[ h'(x) \sim ax^{a-1}, \quad x \to \infty. \]

**Proof.** Let \( h(x) = g(x^a) \). By (1),
\[ | (a - 1)x^{a-2}g'(x^a) + a^2x^{2a-2}g''(x^a) | \leq B | x^{a-1}g'(x^a) |^{(2-a)/(1-a)}, \]
where \( B = Aa^{(2-a)/(1-a)} \). Then
\[ x | g''(x) | \leq B | g'(x) |^b + C | g'(x) |, \]
for suitable constants \( B \) and \( C \). Therefore \( g(x) \) satisfies the conditions of Theorem 3 with
\[ \omega(y) = By^b + Cy, \]
and the proof is complete.

Observe that in the excluded case \( 1 < a < 2 \) the number \( b \) is negative, so that the function \( \omega(y) \) defined by (2) is increasing for sufficiently large \( y \), but not all \( y > 0 \). I have not investigated this case further.

A natural conjecture is to consider the case \( a = \infty \) to correspond to

**Theorem 5.** If
\[ h(x) \sim e^x, \quad x \to \infty, \]
and
\[ | h''(x) | \leq A | h'(x) | \]
then
\[ h'(x) \sim e^x, \quad x \to \infty. \]
Proof. The function $g(x)$ defined by $g(e^x) = h(x)$ satisfies the conditions of Theorem 3.

At the other end of the spectrum it is possible to make several conjectures corresponding to the open case $a = 0, b = 2$ of Theorem 4. First, if $a \neq 0$ the condition $h(x) \sim x^a$ can be replaced by $h(x) \sim a^{-1}(x^a - 1)$ and the conclusion by $h'(x) \sim x^{a-1}$. Letting $a \to 0^+$ suggests the conjecture that the conditions $h(x) \sim \log x$ and

$$\left| h''(x) \right| \leq A \left| h'(x) \right|^2$$

imply

$$h'(x) \sim 1/x, \quad x \to \infty.$$  

This is made plausible by the fact that $\log x$ itself satisfies (3). However, the example $h(x) = \log x + \epsilon \cos(\log x)$ shows the conjecture to be false.

The following theorem holds and is adequate for the applications.

Theorem 6. If

$$\int^x yh(y)dy \sim x, \quad x \to \infty,$$

and (3) is satisfied, then $h'(x) \sim 1/x, x \to \infty$.

For the proof apply Theorem 3 to the function $g(x)$ defined by the integral appearing in (5).

Added in proof (December 19, 1966). The conclusion of Theorem 4 is still valid for $1 < a < 2$ provided (1) is understood to mean

$$\left| h'(x) \right|^{-b} \left| h''(x) \right| \leq A, \quad b = (2 - a)/(1 - a).$$

This follows by the same argument, but substituting for Theorem 3 a theorem of Karamata (Amer. J. Math. 61 (1939), 769–770).

Reference


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