SOME NONLINEAR TAUBERIAN THEOREMS

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In a forthcoming study of the n-body problem I have made use of the nonlinear Tauberian theorems obtained in this note. The symbol \( \omega(x) \) represents a positive function, increasing for \( x > 0 \). The symbols \( f, g, h \) represent functions which are of class \( C^2 \) on \( (0, \infty) \).

The basic Theorem 1 is due to Boas [1].

**Theorem 1.** If

\[
f(x) = o(x), \quad x \to 0 +,
\]

and

\[
f''(x) = \omega(\left| f'(x) \right|)O(x^{-1}), \quad x \to 0 +,
\]

then

\[
f'(x) = o(1), \quad x \to 0 +.
\]

My first theorem asserts that the clause "\( x \to 0 + \)" may be replaced by "\( x \to \infty \)."

**Theorem 2.** If

\[
g(x) = o(x), \quad x \to \infty,
\]

and

\[
g''(x) = \omega(\left| g'(x) \right|)O(x^{-1}), \quad x \to \infty,
\]

then

\[
g'(x) = o(1), \quad x \to \infty.
\]

**Proof.** The function \( f(x) \) defined by

\[
f(x) = x^2g(1/x) - 2 \int_{1/x}^{\infty} (g(u)/u^3)du
\]

satisfies the hypotheses of Theorem 1.

**Theorem 3.** If

\[
g(x) \sim x, \quad x \to \infty,
\]

and

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\[ g''(x) = \omega(\left| g'(x) \right|)O(x^{-1}), \quad x \to \infty, \]

then

\[ g'(x) \sim 1, \quad x \to \infty. \]

**Proof.** The function \( h(x) = g(x) - x \) satisfies the conditions of Theorem 2 with \( \omega(y) \) replaced by \( \omega(y + 1) \).

**Theorem 4.** Let \( a \) be a number satisfying \( 0 < a < 1 \), or \( 2 \leq a < \infty \). If

\[ h(x) \sim x^a, \quad x \to \infty, \]

and

\[ (1) \quad \left| h''(x) \right| \leq A \left| h'(x) \right|^b, \quad b = (2 - a)/(1 - a), \]

then

\[ h'(x) \sim ax^{a-1}, \quad x \to \infty. \]

**Proof.** Let \( h(x) = g(x^a) \). By (1),

\[ \left| (a - 1)x^{a-2}g'(x^a) + a^2x^{2a-2}g''(x^a) \right| \leq B \left| x^{a-1}g'(x^a) \right|^{(2-a)/(1-a)}, \]

where \( B = Aa^{(2-a)/(1-a)} \). Then

\[ x \left| g''(x) \right| \leq B \left| g'(x) \right|^b + C \left| g'(x) \right|, \]

for suitable constants \( B \) and \( C \). Therefore \( g(x) \) satisfies the conditions of Theorem 3 with

\[ (2) \quad \omega(y) = By^b + Cy, \]

and the proof is complete.

Observe that in the excluded case \( 1 < a < 2 \) the number \( b \) is negative, so that the function \( \omega(y) \) defined by (2) is increasing for sufficiently large \( y \), but not all \( y > 0 \). I have not investigated this case further.

A natural conjecture is to consider the case \( a = \infty \) to correspond to

**Theorem 5.** If

\[ h(x) \sim e^x, \quad x \to \infty, \]

and

\[ \left| h''(x) \right| \leq A \left| h'(x) \right| \]

then

\[ h'(x) \sim e^x, \quad x \to \infty. \]
Proof. The function \( g(x) \) defined by \( g(e^x) = h(x) \) satisfies the conditions of Theorem 3.

At the other end of the spectrum it is possible to make several conjectures corresponding to the open case \( a = 0, b = 2 \) of Theorem 4. First, if \( a \neq 0 \) the condition \( h(x) \sim x^a \) can be replaced by \( h(x) \sim a^{-1}(x^a - 1) \) and the conclusion by \( h'(x) \sim x^{a-1} \). Letting \( a \to 0+ \) suggests the conjecture that the conditions \( h(x) \sim \log x \) and

\[
\left| h''(x) \right| \leq A \left| h'(x) \right|^2
\]

imply

\[
h'(x) \sim \frac{1}{x}, \quad x \to \infty.
\]

This is made plausible by the fact that \( \log x \) itself satisfies (3). However, the example \( h(x) = \log x + \epsilon \cos(\log x) \) shows the conjecture to be false.

The following theorem holds and is adequate for the applications.

**Theorem 6.** If

\[
\int^x y h'(y) dy \sim x, \quad x \to \infty,
\]

and (3) is satisfied, then \( h'(x) \sim 1/x, \quad x \to \infty \).

For the proof apply Theorem 3 to the function \( g(x) \) defined by the integral appearing in (5).

**Added in proof (December 19, 1966).** The conclusion of Theorem 4 is still valid for \( 1 < a < 2 \) provided (1) is understood to mean

\[
| h'(x) |^{-b} | h''(x) | \leq A, \quad b = (2 - a)/(1 - a).
\]

This follows by the same argument, but substituting for Theorem 3 a theorem of Karamata (Amer. J. Math. 61 (1939), 769–770).

**Reference**


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