STEENROD REPRESENTABILITY OF STABLE HOMOLOGY

PETER S. LANDWEBER

1. **Introduction.** N. Steenrod posed the problem of determining those homology classes of a space which can be represented as the image of the fundamental class of a manifold. The problem, which is central in cobordism theory, has since been treated by R. Thom [**12**], P. Conner and E. Floyd [**6**], [**7**], and Y. Shikata [**11**], among others. In this paper the manifolds will be weakly almost complex ($\mathfrak{U}$-manifolds). A solution will be obtained in the stable range for the Eilenberg-MacLane spaces $K(\pi, q)$, π a cyclic group of prime or infinite order, and for the spaces $BU[2q]$ fibred over BU with $\pi_i(BU[2q])=0$ when $i<2q$ and $\pi_i(BU[2q])\to\pi_i(BU)$ an isomorphism when $i\geqq 2q$. The spaces $BU[2q]$ were studied by J. F. Adams [**1**], who denoted them $BU(2q, \cdots, \infty)$.

We now introduce some notation. All spaces will have base points, and all homology and cohomology theories will be reduced. For p a prime, ρ_p: $\tilde{H}_*(X; Z)\to\tilde{H}_*(X; Z_p)$ denotes the usual coefficient homomorphism. $\tilde{\mathfrak{U}}_*(X)=\tilde{H}_*(X;\ MU)$ is the (graded) complex bordism group of X, i.e., the homology of X with coefficients in the Milnor spectrum MU [**7**], [**10**]. There is a natural transformation μ: $\tilde{\mathfrak{U}}_*(\ \ )\to\tilde{H}_*(\ \ ;\ Z)$ of homology theories; we denote by μ_p: $\tilde{\mathfrak{U}}_*(X)\to\tilde{H}_*(X; Z_p)$ the composition $\rho_p\mu$. In terms of μ, the problem posed above becomes that of determining the image of μ in $\tilde{H}_*(X; Z)$; see [**6**] for the geometric interpretation of μ.

The main results, along with several of an auxiliary nature, are summarized below. Their proofs are given in §§3 and 4. Some observations on the homology of a spectrum, with coefficients in another spectrum, are collected in §2.

LEMMA (1.1). *For any of the spaces $K(\pi, q)$, π cyclic of prime or infinite order, or $BU[2q]$, $\bigcap_p$ ker $(\rho_p)=0$ in the stable range.*

THEOREM (1.2). *For any of the spaces $K(\pi, q)$, π cyclic of prime or infinite order or $BU[2q]$, a homology class x in the stable range is in the image of μ if and only if $\rho_p(x)$ is in the image of μ_p for all primes p.*

We are therefore led to consider the homomorphisms μ_p: $\tilde{\mathfrak{U}}_*(X)\to\tilde{H}_*(X; Z_p)$. Let $A^*(p)$ denote the mod p Steenrod algebra, and let $\chi: A^*(p)\to A^*(p)$ denote the canonical anti-automorphism [**9**]. Given $a\in A^i(p)$, make a act on $\tilde{H}_*(X; Z_p)$ by means of the Kronecker index:

Received by the editors July 4, 1966.

$$\langle ax, \xi \rangle = \langle x, \chi(a)\xi \rangle$$

for $x \in \tilde{H}_k(X; Z_p)$ and $\xi \in \tilde{H}^{k-i}(X; Z_p)$; thus a lowers degrees by i in homology. Let $I^*(p)$ denote the two-sided ideal in $A^*(p)$ generated by the Bockstein $Q_0^{(p)}$.

LEMMA (1.3). *For any space with base point, the image of μ_p is annihilated by all (homology) operations in $I^*(p)$.*

THEOREM (1.4). *For X any of the spaces $K(\pi, q)$, Π cyclic of prime or infinite order, or $BU[2q]$, the image of μ_p in the stable range is exactly the subspace of $\tilde{H}_*(X; Z_p)$ annihlated by $I^*(p)$.*

The following result may be of independent interest. Adams has defined classes $ch_{q,r} \in H^{2q+2r}(BU[2q]; Z)$ for $q \geqq 0$ and $r \geqq 0$.

PROPOSITION (1.5). *For $r<q$, one can construct $y_{q,r} \in \tilde{\mathfrak{U}}_{2q+2r}(BU[2q])$ so that $\langle \mu(y_{q,r}), ch_{q,r} \rangle = 1$.*

2. **Homology of a spectrum.** The families of spaces $K(\pi, q)$ and $BU[2q]$, as well as the Thom spaces $MU(q)$, all yield spectra. We denote these spectra by $K(\pi)$, bu and MU. (The spectrum bu has spaces $bu_{2q} = BU[2q]$, $bu_{2q-1} = \Omega BU[2q]$; MU has spaces $MU_{2q} = MU(q)$, $MU_{2q+1} = S^1 \wedge MU(q)$.) If $\boldsymbol{M} = [M_q]$ denotes one of these spectra, then $\boldsymbol{M}$ has the following properties:

(a) M_q is $(q-1)$-connected;

(b) $SM_q \to M_{q+1}$ induces isomorphisms $\pi_i(SM_q) \to \pi_i(M_{q+1})$ for $i < 2q$ and an epimorphism for $i = 2q$.

We want to be able to prove results of the following sorts: $\tilde{\mathfrak{U}}_{2q+2r}(BU[2q])$ is free abelian for $r<q$; $\tilde{\mathfrak{U}}_{2q+2r+1}(BU[2q]) = 0$ if $r < q-1$; the map $MU(q) \to BU[2q]$ obtained from the K-theory Thom class induces an epimorphism $\tilde{\mathfrak{U}}_{2q+2r}(MU(q)) \to \tilde{\mathfrak{U}}_{2q+2r}(BU[2q])$ for $r<q$. Such results may be proved by letting $q \to \infty$ to obtain homology groups of a spectrum with coefficients in another spectrum —e.g., $H_r(bu; MU)$—and by then noticing that we may interchange the roles of the two spectra and re-examine the questions. E.g., $H_r(bu; MU) \approx H_r(MU; bu)$, and the latter group may be investigated with the help of the homology theory $\tilde{H}_*(\ \ ; bu)$. These ideas are due to A. Hattori [8].

Let $\boldsymbol{M} = [M_q]$ and $\boldsymbol{M}' = [M_s']$ be two spectra satisfying (a) and (b) above. Then

$$\tilde{H}_{q+r}(M_q; \boldsymbol{M}') = \lim_{s \to \infty} \pi_{q+r+s}(M_q \wedge M_s').$$

We define $H_r(\boldsymbol{M}; \boldsymbol{M}') = \lim_{q \to \infty} \tilde{H}_{q+r}(M_q; \boldsymbol{M}')$, so that

$$H_r(\boldsymbol{M};\boldsymbol{M}') = \lim_{q\to\infty} \lim_{s\to\infty} \pi_{q+r+s}(M_q \wedge M'_s).$$

Taking the limit in the other order and twisting the factors, we obtain $H_r(\boldsymbol{M}'; \boldsymbol{M})$, proving

LEMMA (2.1). $H_r(\boldsymbol{M}; \boldsymbol{M}') \approx H_r(\boldsymbol{M}', \boldsymbol{M})$.

LEMMA. (2.2). *If* $r<q-1$, *then*

$$\tilde{H}_{q+r}(M_q;\boldsymbol{M}') \approx H_r(\boldsymbol{M};\boldsymbol{M}').$$

PROOF. It suffices to show that $\tilde{H}_{q+r}(M_q;\, \boldsymbol{M}')\to\tilde{H}_{q+r+1}(M_{q+1};\, \boldsymbol{M}')$ is an isomorphism for $r<q-1$, and for this we must show that

$$\tilde{H}_{q+r+1}(SM_q;\boldsymbol{M}') \to \tilde{H}_{q+r+1}(M_{q+1};\boldsymbol{M}')$$

is an isomorphism for $r<q-1$. Regard $SM_q\to M_{q+1}$ as an inclusion, and put $N_{q+1}=M_{q+1}/SM_q$. Since $\boldsymbol{M}$ satisfies (a), all three spaces SM_q, M_{q+1}, N_{q+1} are q-connected. It follows from the generalized suspension theorem [3] that for any of these spaces $\pi_i(\ \ )=\pi_i^s(\ \ )$ if $i\leqq 2q(\pi_i^s(\ \ )=\tilde{H}_i(\ \ ;\ \boldsymbol{S})$ is stable homotopy, where $\boldsymbol{S}$ denotes the sphere spectrum). Since $\boldsymbol{M}$ satisfies (b), the exact sequence for stable homotopy implies that $\pi_i^s(N_{q+1})=0$ if $i\leqq 2q$. Hence also $\pi_i(N_{q+1})=0$ for $i\leqq 2q$, so N_{q+1} is $2q$-connected. Then the spectral sequence for $\tilde{H}_*(N_{q+1};\, \boldsymbol{M}')$ shows that $\tilde{H}_i(N_{q+1};\, \boldsymbol{M}')=0$ if $i\leqq 2q$, since $\boldsymbol{M}'$ satisfies (a). From the exact sequence for $\tilde{H}_*(\ \ ;\, \boldsymbol{M}')$, it follows that $\tilde{H}_i(SM_q;\, \boldsymbol{M}')\to\tilde{H}_i(M_{q+1};\, \boldsymbol{M}')$ is an isomorphism if $i<2q$ (and onto if $i=2q$). Q.E.D.

The spectra mentioned at the beginning of this section are multiplicative; this is well known for $K(\pi)$ and MU, and has been shown for bu by D. Anderson [2]. The coefficient ring $\tilde{H}_*(S^0;\, bu)$ is a polynomial ring over the integers on a 2-dimensional generator, and so has no torsion. The maps $MU(q)\to BU[2q]$ mentioned above give rise to a multiplicative map of spectra $\boldsymbol{\nu}$: $MU\to bu$ ([2]); this induces an epimorphism of coefficient rings $\tilde{H}_*(S^0;\, MU)\to\tilde{H}_*(S^0;\, \boldsymbol{bu})$, which is essentially the Todd genus. Moreover, there are maps $BU[2q]\to K(Z,\, 2q)$ corresponding to $ch_{q,0}\in H^{2q}(BU[2q];\, Z)$, which give rise to another multiplicative map of spectra $\boldsymbol{\lambda}: bu\to K(Z)$. The composition $MU\to bu\to K(Z)$ is the usual map $\boldsymbol{\mu}: MU\to K(Z)$ resulting from the cohomology Thom class; this latter map induces the natural transformation μ. Let $\boldsymbol{\nu}$ and $\boldsymbol{\lambda}$ induce ν, λ respectively.

LEMMA (2.3). *Let X be a CW-complex with base point with $\tilde{H}_*(X;\, Z)$ free abelian. Then $\tilde{H}_*(X;\, MU)$ and $\tilde{H}_*(X;\, bu)$ are free abelian and the homomorphisms*

$$\tilde{H}_*(X; MU) \xrightarrow{\nu} \tilde{H}_*(X; bu),\ \tilde{H}_*(X; bu) \xrightarrow{\lambda} \tilde{H}_*(X; Z)$$

are epimorphisms.

PROOF. The spectral sequences for $\tilde{H}_*(X;\ MU)$ and $\tilde{H}_*(X;\ bu)$ both collapse, since $\tilde{H}_*(X;\ Z)$ and both coefficient rings are free abelian. Thus both $\mu: \tilde{H}_*(X;\ MU) \to \tilde{H}_*(X;\ Z)$ and λ map onto $\tilde{H}_*(X;\ Z)$. Let (c_i) be a homogeneous basis for $\tilde{H}_*(X;\ Z)$ as a free abelian group. Select a homogeneous pre-image γ_i of c_i in $\tilde{H}_*(X;\ MU)$; then the image γ_i' of γ_i in $\tilde{H}_*(X;\ bu)$ is also a pre-image of c_i. It follows easily, from an argument used by P. Conner and E. Floyd [6], that $\tilde{H}_*(X;\ MU)$ is a free $\tilde{H}_*(S^0;\ MU)$-module on the γ_i, and that $\tilde{H}_*(X;\ bu)$ is a free $\tilde{H}_*(S^0;\ bu)$-module on the γ_i'. Thus all assertions are proved. Q.E.D.

As a consequence of these lemmas, we obtain commutative diagrams such as

$$\begin{array}{ccc} H_*(MU; MU) & \xrightarrow{\text{epi}} & H_*(MU; Z) \\ \big\downarrow \text{epi} & & \big\downarrow \nu_* \\ H_*(bu; MU) & \xrightarrow{\mu} & H_*(bu; Z) \end{array} \tag{2.4}$$

showing that $\mathrm{im}(\mu) = \mathrm{im}(\nu_*)$ in $H_*(bu;\ Z)$. There is a similar diagram if bu is replaced by $K(\pi)$, π cyclic, and also diagrams such as

$$\begin{array}{ccc} H_*(MU; MU) & \xrightarrow{\text{epi}} & H_*(MU; Z_p) \\ \big\downarrow \text{epi} & & \big\downarrow \nu_* \\ H_*(bu; MU) & \xrightarrow{\mu_p} & H_*(bu; Z_p) \end{array} \tag{2.5}$$

showing that $\mathrm{im}(\mu_p) = \mathrm{im}(\nu_*)$ in $H_*(bu;\ Z_p)$.

3. **Proof of Theorem (1.2).** To begin, we remark that for any of the spaces $K(\pi,\ q)$, π cyclic of prime or infinite order, or $BU[2q]$, the cohomology in the stable range has no elements of order p^2 for any prime p. This is shown by Adams [1] in the latter case, and is well known in the other cases. By the universal coefficient theorem the stable homology of these spaces has no elements of order p^2. Since the homology groups are finitely generated, Lemma (1.1) follows immediately.

Theorem (1.2) is immediate for $K(\pi,\ q)$ if π is a finite cyclic group of prime order. With $\pi = Z$, the Hurewicz theorem implies that, in dimension q, μ and all μ_p are onto. Thus we confine attention to dimensions $> q$. Let $x \in \tilde{H}_{q+r}(K(Z,\ q);\ Z)$ with $\rho_p(x) = \mu_p(y_p)$ for all primes p. Since $\tilde{H}_{q+r}(K(Z,\ q);\ Z_p) = 0$ for almost all primes p, we may assume that almost all the y_p vanish. By elementary number theory, if we replace y_p by a suitable multiple we may also assume that $\mu_{p'}(y_p)=0$ if $p'\neq p$. Then putting $y=\sum y_p$, we have $\rho_p(x)=\mu_p(y)$ for all, p, i.e., $\rho_p(x-\mu(y))=0$ for all p, so by Lemma (1.1) $x=\mu(y)$. Thus Theorem (1.2) is proved in this case.

Finally we consider $BU[2q]$, first proving Proposition (1.5). Recall that the classes $\mathrm{ch}_{q,r}\in H^{2q+2r}(BU[2q];Z)$ defined by Adams [1] pass in rational cohomology to $m(r)\mathrm{ch}_{q+r}$, where

$$m(r) = \prod_p p^{[r/(p-1)]}$$

and ch_{q+r} is the component of the Chern character in dimension $2q+2r$. We wish to construct a class $y_{q,r}\in\tilde{\mathfrak{U}}_{2q+2r}(BU[2q])$ so that $\langle\mu(y_{q,r}),\mathrm{ch}_{q,r}\rangle=1$, provided that $r<q$. That is, we seek a $\mathfrak{U}$-manifold M^{2q+2r} and $\alpha\in K(M)$ so that $[M]=0$ in the complex bordism ring and α is trivial on the $(2q-1)$-skeleton of M, for which $\langle m(r)\mathrm{ch}_{q+r}(\alpha), M\rangle=1$.

Let $\omega=(i_1,\cdots,i_k)$ be a partition of degree r, so that $k\leqq r<q$. Put $M_\omega=CP(i_1+1)\times\cdots\times CP(i_k+1)\times S^{2q-2k}$ and let $\alpha_\omega\in K(M_\omega)$ be the product of the bundles $\eta-1$ on the projective spaces (η the Hopf bundle) and the generating bundle of $\tilde{K}(S^{2q-2k})$. Then M_ω is a $\mathfrak{U}$-manifold of dimension $2q+2r$, $[M_\omega]=0$ in the complex bordism ring, and α_ω is trivial on the $(2q-1)$-skeleton of M_ω. Thus a classifying map for α_ω lifts to a map $M_\omega\to BU[2q]$, and so we obtain $y_\omega\in\tilde{\mathfrak{U}}_{2q+2r}(BU[2q])$. Put $(\omega+1)!=(i_1+1)!\cdots(i_k+1)!$; it is easily seen that

$$\langle\mu(y_\omega),\mathrm{ch}_{q,r}\rangle = m(r)/(\omega+1)!$$

LEMMA (3.1). *The integers $m(r)/(\omega+1)!$, as ω runs through the partitions of degree r, have greatest common divisor* 1.

PROOF. It suffices to exhibit for each prime $p<r$ a partition ω of degree r with $m(r)/(\omega+1)!$ prime to p, since $m(r)$ is divisible by no greater primes. Given p, let ω be any partition of degree r with $p-1$ occurring exactly $[r/(p-1)]$ times. Then $p[r/(p-1)]$ divides $(\omega+1)!$, and so $m(r)/(\omega+1)!$ is prime to p. Q.E.D.

PROOF OF (1.5). For each partition ω of degree r, we have constructed y_ω in $\tilde{\mathfrak{U}}_{2q+2r}(BU[2q])$ with $\langle\mu(y_\omega),\mathrm{ch}_{q,r}\rangle=m(r)/(\omega+1)!$. According to the lemma just established, there is an integral linear combination $y_{q,r}=\sum a_\omega y_\omega$ with $\langle\mu(y_{q,r}),\mathrm{ch}_{q,r}\rangle=1$. Q.E.D.

We now prove Theorem (1.2) for $BU[2q]$. Concerning the cohomology of $BU[2q]$, the following remarks are needed (see [1]). For r odd, $r<2q-2$, $H^{2q+r}(BU[2q];Z_p)=0$ for almost all primes; for

$r<q$, $H^{2q+2r}(BU[2q]; Z_p)\approx Z_p$ with generator $\rho_p(\mathrm{ch}_{q,r})$ for almost all primes p. Now let $x\in\tilde{H}_*(BU[2q]; Z)$ be given in the stable range with $\rho_p(x)\in\mathrm{im}(\mu_p)$ for all primes p. If x has odd degree, we may argue as above that $x\in\mathrm{im}(\mu)$. Thus suppose $x\in H_{2q+2r}(BU[2q]; Z)$; by virtue of Proposition (1.5), we may also assume that $\langle x, \mathrm{ch}_{q,r}\rangle=0$. Hence $\rho_p(x)=0$ for almost all primes p. Now say $\rho_p(x)=\mu_p(y_p)$ for all primes p, with almost all the y_p zero. Again by Proposition (1.5), we may assume that $\langle\mu(y_p), \mathrm{ch}_{q,r}\rangle=0$ for all p, so that $\mu_{p'}(y_p)=0$ for almost all primes p'. Replacing the y_p by suitable multiples, we may further require that $\mu_{p'}(y_p)=0$ for all prime p, p' ($p\neq p'$). Putting $y=\sum y_p$, we see that $\rho_p(x)=\mu_p(y)$ for all primes p, so that $x=\mu(y)$ by Lemma (1.1). This completes the proof.

4. **Proof of Theorem (1.4).** We begin by proving Lemma (1.3). Recall that the mod p Steenrod algebra $A^*(p)$ acts on $\tilde{H}_*(X; Z_p)$ via the Kronecker index. The two-sided ideal $I^*(p)$ in $A^*(p)$ generated by the Bockstein $Q_0^{(p)}$ is also the left (or right) ideal generated by the elements of odd degree, according to Milnor [**10**].

Proof of (1.3). In view of the above remark, it suffices to show that any stable operation $\tilde{\mathfrak{U}}_*(\ )\to\tilde{H}_*(\ ; Z_p)$ of odd degree is trivial. By Alexander-Spanier duality [**13**], such an operation corresponds to a stable operation $\tilde{\mathfrak{U}}^*(\ )\to\tilde{H}^*(\ ; Z_p)$, and so to an element of $\tilde{H}^*(MU; Z_p)$. Since the stable cohomology of MU vanishes in odd dimensions, the result follows. Q.E.D.

Theorem (1.4) will now be proved for $BU[2q]$; a simplified, argument suffices to treat $K(\pi, q)$, π cyclic of prime or infinite order. It follows from (2.5) that in the stable range $\mathrm{im}(\mu_p)=\mathrm{im}(\nu_*)$ in $\tilde{H}_*(BU[2q]; Z_p)$, where $\nu_*:\tilde{H}_*(MU(q); Z_p)\to\tilde{H}_*(BU[2q]; Z_p)$ is induced by the map ν: $MU(q)\to BU[2q]$. There is also an induced map in cohomology $\nu^*:\tilde{H}^*(BU[2q]; Z_p)\to\tilde{H}^*(MU(q); Z_p)$. Making use of the Kronecker index, it suffices to show that in the stable range

$$\ker(\nu^*) = I^*(p)\cdot\tilde{H}^*(BU[2q]; Z_p). \tag{4.1}$$

According to Adams [**1**], $\tilde{H}^*(BU[2q]; Z_p)$ is isomorphic in the stable range to a direct sum of $A^*(p)$-modules isomorphic to $A^*(p)/J^*(p)$ on generators $\rho_p(\mathrm{ch}_{q,r})$ with $r=0, 1, \cdots, p-2$; here $J^*(p)$ is the left ideal generated by $Q_0^{(p)}\in A^1(p)$ and $Q_1^{(p)}\in A^{2p-1}(p)$. Milnor [**10**] has shown that $\tilde{H}^*(MU(q); Z_p)$ is isomorphic in the stable range to a direct sum of free modules over $A^*(p)/I^*(p)$, on generators s_ω of degree 2 deg ω, where ω is any partition containing no integer of the form p^i-1. Thus elements of $\tilde{H}^*(MU(q); Z_p)$ of degree$\leqq 2q+2(p-2)$ are independent over $A^*(p)/I^*(p)$ if they are independent over Z_p.

Let $\phi: H^*(BU(q)) \to \tilde{H}^*(MU(q))$ denote the Thom isomorphism, and let ζ_q be the universal bundle over $BU(q)$. If u_q denotes the K-theory Thom class of ζ_q, then $\nu^*(\mathrm{ch}_{q,r}) = \mathrm{ch}_{q,r}(u_q) = m(r)\mathrm{ch}_{q+r}(u_q)$; we regard $\tilde{H}^*(MU(q); Z) \subset \tilde{H}^*(MU(q); Q)$ as usual. Now $\mathrm{ch}_{q+r}(u_q) = \phi(T_r(-\zeta_q))$ according to Bott [5], where $T_r(-\zeta_q)$ is the component of degree $2r$ of the Todd class of the "negative" of the bundle ζ_q. Therefore $\nu^*(\mathrm{ch}_{q+r}) = \phi(m(r)T_r(-\zeta_q))$. It has been shown by M. Atiyah and F. Hirzebruch [4] that $m(r)T_r(-\zeta_q) \in H^{2r}(BU(q); Z)$ is divisible by no primes, so that for p a prime $\rho_p[m(r)T_r(-\zeta_q)] \neq 0$. Hence $\nu^*[\rho_p(\mathrm{ch}_{q,r})] \neq 0$ for $r = 0, 1, \cdots, p-2$, so these elements of $\tilde{H}^*(MU(q); Z_p)$ are independent over $A^*(p)/I^*(p)$ according to the preceding paragraph. Thus (4.1) is proved, and with it Theorem (1.4) for $BU[2q]$. Q.E.D.

REMARK. The image of $\mu_p: \mathfrak{U}_*(K(Z_p)) \to H_*(K(Z_p); Z_p)$ may be best described with the help of the dual Hopf algebra $A_*(p)$ to $A^*(p)$. Milnor has shown [9] that $A_*(p)$ is a tensor product of an exterior subalgebra with a polynomial subalgebra. If we identify $A_*(p)$ with $H_*(K(Z_p); Z_p)$, the image of μ_p is exactly the polynomial subalgebra which is the annihilator of the ideal $I^*(p)$ in $A^*(p)$.

REFERENCES

1. J. F. Adams, *On Chern characters and the structure of the unitary group*, Proc. Cambridge Philos. Soc. **57** (1961), 189–199.

2. D. Anderson, unpublished.

3. M. Atiyah, *Bordism and cobordism*, Proc. Cambridge Philos. Soc. **57** (1961), 200–208.

4. M. Atiyah and F. Hirzebruch, *Cohomologie-Operationen und charakteristische Klassen*, Math. Z. **77** (1961), 149–187.

5. R. Bott, *Lectures on* $K(X)$, (mimeographed), Harvard Univ., Cambridge, Mass., 1962.

6. P. Conner and E. Floyd, *Differentiable periodic maps*, Springer, Berlin, 1964.

7. ———, *Torsion in SU-bordism*, Mem. Amer. Math. Soc. No. 60 (1966), 74 bb.

8. A. Hattori, *Integral characteristic numbers for weakly almost complex manifolds*, Topology **5** (1966), 259–280.

9. J. Milnor, *The Steenrod algebra and its dual*, Ann. of Math. **67** (1958), 150–171.

10. ———, *On the cobardism ring* Ω^* *and a complex analogue*. I, Amer. J. Math. **82** (1960), 505–521.

11. Y. Shikata, *On the realizabilities of integral homology classes by orientable submanifolds*, Osaka J. Math. **12** (1961), 79–87.

12. R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954), 17–86.

13. G. Whitehead, *Generalized homology theories*, Trans. Amer. Math. Soc. **102** (1962), 227–283.

UNIVERSITY OF VIRGINIA